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COHERENCY AND REGULARITY OF DEMAND SYSTEMS
WITH EQUALITY AND INEQUALITY CONSTRAINTS

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COHERENCY AND REGULARITY OF DEMAND SYSTEMS
WITH EQUALITY AND INEQUALITY CONSTRAINTS

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ABSTRACT

In models dealing with rationing, corner solutions, non-linear budget constraints, or endogenously switching regimes, utility theory plays a more crucial role than in traditional demand systems. If in these models negativity of the Slutsky matrix is violated, the models will in general not be coherent, in the sense that the endogenous variables are not determined unambiguously by the model, or, loosely speaking, that 'probabilities' do not sum to one. The first goal of this paper is to illustrate the type of coherency problems that may be encountered in some non-standard neoclassical models. Secondly, the paper points out that estimating without imposing coherency is inappropriate in the sense that it will often yield inconsistent estimators, even though the true data generating process is coherent. In the third place, the paper shows how to impose neoclassical regularity conditions in some 'large enough' region of quantity and/or price space. Imposing these restrictions is in general sufficient for coherency, and we argue that in many cases it is also 'almost necessary'.

1. Introduction

Empirical researchers in the field of demand theory are becoming increasingly aware of the tight structure that may be imposed on their models by neoclassical theory. In the somewhat older literature on demand systems a typical approach would be to choose a particular representation of preferences and derive the corresponding demand functions. After tacking on an error term, the system would next be estimated. In the estimation, restrictions from neoclassical theory might or might not be imposed. In either case authors often have tested the various Slutsky conditions for their particular empirical specification, with mixed results (see, e.g., Barten, 1977). As noted by McElroy (1987) the attention for consistency with neoclassical theory has mostly been limited to the systematic part of the demand equations, with a rather cavalier treatment of the error structure. Her own work is a notable exception in this respect.

Whether or not authors would severely test neoclassical restrictions for their data set, it seems fair to say that in a standard demand system the empirical specification is rather loosely connected with the underlying theory. If the estimation results turn out to be inconsistent with a utility maximization hypothesis, the empirical model can still be regarded as an adequate description of reality.

This is no longer true in more complicated situations where the theory is used more intensively. In models dealing with rationing, corner solutions, non-linear budget constraints, or endogenously switching regimes, utility theory plays a more crucial role than in traditional demand systems. If in these models regularity conditions are violated, then these models will in general not be coherent, in the sense that the endogenous variable is not determined unambiguously by the model, or, in other words, the reduced form is not well-defined. See e.g. Heckman (1978), who refers to coherency as 'the principal assumption'.

The first goal of this paper is to illustrate the type of coherency problems that may be encountered in non-standard neoclassical models by means of two examples (section 2). In the literature, coherency of some specific models has been analysed before. For example, Ransom (1987a) has noted that the demand system based on the quadratic utility function with non-negativity constraints introduced by Wales and Woodland (1983) is coherent if the parameters satisfy certain regularity conditions, which are closely connected to global concavity of the corresponding expenditure

function (i.e., 'negativity'). Similarly, Hausman (1985) and MaCurdy et al. (1988) note the importance of negativity for coherency in individual labour supply models with kinked budget constraints. In Van Soest and Kooreman (1987), it is shown that the approach of Lee and Pitt (1986) to the indirect translog demand system with non-negativity constraints via the use of shadow prices may lead to incoherent models unless conditions are imposed on the parameters. These conditions appear to be closely related to concavity of the expenditure function. In the context of simultaneous linear equation models with endogenously switching regimes, coherency has been analysed by Gourieroux et al. (1980). In the models they consider, coherency conditions have the form of restrictions on parameters only. In the models that we consider, conditions generally not only depend on fixed parameters, but also on the possible values of the exogenous variables and of the error terms, in particular those representing random preferences.

In empirical applications, one possible approach is to ignore coherency conditions in estimating the model and check afterwards (per observation) whether coherency or regularity conditions are satisfied. The second goal of this paper is to point out that this practice is inappropriate, in the sense that it may lead to inconsistent estimators, at least if maximum likelihood is used. Let Θ be the space of parameters which generate coherent models and let $\Theta^* \supset \Theta$ be an extension of the parameter space, including parameters which do not yield a coherent model. In section 3, we present an example of a bivariate Probit model, for which the likelihood function (which is defined on Θ only) can be written as a product of cumulative normal probabilities. The same formula can be used to define a natural extension of the likelihood function to Θ^* . If coherency is ignored, this extension will be maximized on Θ^* . We show that this yields an inconsistent estimator with probability limit outside Θ , even though the true parameter vector belongs to Θ . The intuitive explanation is that the sum of 'probabilities' of events which are mutually exclusive if the model is coherent, may exceed one for parameter values outside Θ .

Given the fact that without imposing coherency ML estimation is inappropriate, the first problem in empirical work is to formulate necessary and sufficient conditions for coherency in a particular model. In a well-defined neoclassical model, imposition of all the regularity conditions from demand theory is sufficient for coherency. The vast majority of the specifications of preference structures considered in the

literature only satisfy regularity conditions locally, i.e. in some subset of quantity or price space. For most flexible systems, the relationship between the parameter values and this subset is far from obvious. See, e.g., Barnett and Lee (1985) and Barnett (1983). For other systems, such as the generalized McFadden cost function proposed by Diewert and Wales (1987), explicit expressions of demand functions or conditional demand functions (necessary in case of binding constraints) cannot be obtained.

We will focus on how to impose the regularity conditions in some 'large enough' region of quantity or price space. We discuss this first for the case of a standard demand system, an inverse demand system, and a conditional demand system (section 4). The regularity conditions imply that the set of feasible (fixed and random) parameters cannot be too large. On the other hand, an extra condition is introduced which implies that the parameter space must be large enough, since the model must be able to explain certain features of the data. This is particularly relevant if measurement or optimization errors are excluded. In section 5 we show how the conditions can be imposed for some frequently used specifications of preferences.

In general, regularity conditions for the models in section 4 are necessary to guarantee the micro-economic foundation of the model. Coherency however is hardly a problem. The only issue is whether the domain of the demand functions is large enough. This is different in the models in section 6, which are characterized by endogenously switching regimes. Due to non-linearities, the coherency conditions of Gourieroux et al. (1980) do not apply. We propose to impose regularity conditions quite similar to those introduced in section 4, which are sufficient for coherency. We illustrate this procedure with some examples. Concluding remarks are mentioned in section 7.

2. Concavity and Coherency; two examples

Example 1. A labour supply model with kinked budget constraints

Figure 1 illustrates the simplest possible case of a standard model of individual labour supply in the presence of kinked budget constraints, as developed by Hausman in numerous papers (see, e.g., Hausman, 1981, 1985). Given the budget constraint, the individual chooses the number of hours which maximizes utility (h^* in the figure).

The individual is thus assumed to solve the problem

$$\text{Max}_{h,c} U(h,c) \quad \text{s.t.} \quad h \geq 0, \quad c \leq w_1 h + y_1, \quad c \leq w_2 h + y_2, \quad \text{and} \quad h \leq T. \quad (1)$$

Here h is the number of hours worked per period and c is total consumption or income. A specification of the direct utility function $U(h,c)$ often used in this kind of work is (see, e.g., Hausman, 1981, Blomquist, 1983)

$$U(h,c) = (\delta h - \beta) \exp\{\delta(h - \gamma - \delta c) / (\beta - \delta h)\}, \quad (2)$$

U is increasing in c if $\delta \neq 0$. Along each linear segment of the budget curve, this utility function implies linear labour supply functions:

$$h_i = \beta w_i + \delta y_i + \gamma \quad (i=1,2), \quad (3)$$

where w_i is (minus) the slope of the i -th segment, y_i is the intercept of the i -th segment with the line $h=0$, and h_i is the desired number of hours if the wage rate is w_i and non-labour income is y_i . We assume $0 < w_2 < w_1$, $y_1 < y_2$, and $0 < h_0 < T$, where (h_0, c_0) is the intersection of the two segments.

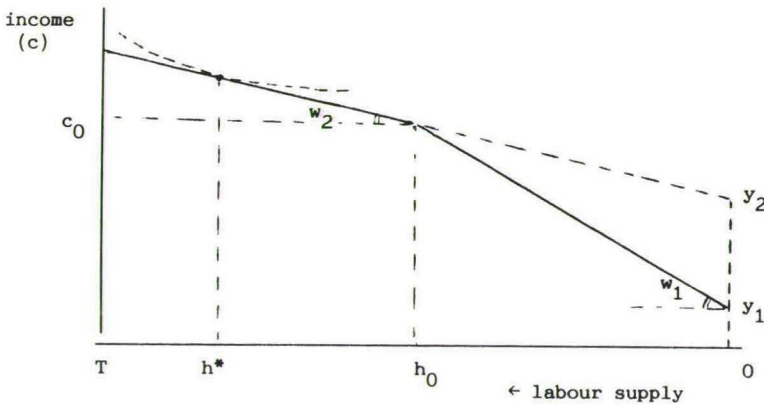


Figure 1. Individual labour supply and a kinked budget constraint

It is straightforward to show that the direct utility function given by (2) is strictly quasi-concave at (h,c) if and only if

$$\beta - \delta h > 0. \quad (4)$$

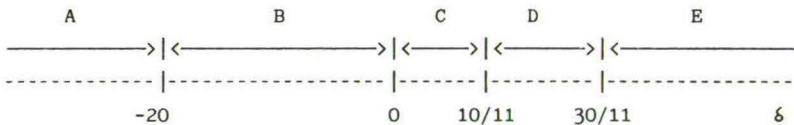
If the direct utility function is strictly quasi-concave on the whole budget set, then Lagrange theory can be applied and the optimum h^* can easily be found with (3). There are five possibilities:

$$\begin{array}{ll}
 \text{A. } h_1 \leq 0 & h^* = 0 \\
 \text{B. } 0 < h_1 \leq h_0 & h^* = h_1 \\
 \text{C. } h_2 \leq h_0 \leq h_1 & h^* = h_0 \\
 \text{D. } h_0 < h_2 \leq T & h^* = h_2 \\
 \text{E. } T < h_2 & h^* = T.
 \end{array} \tag{5}$$

To allow for unobserved preference variation, we assume that δ is a random variable defined on the real line. Hausman (1981) and Blomquist (1983) assume that δ is negative with probability one. For $\beta > 0$ this guarantees quasi-concavity of U at all points of the budget set. In this example we show what can happen if concavity is neglected and δ is allowed to be positive, but nevertheless (5) is applied. Let us assume that the probability distribution of δ is absolute continuous with support \mathbb{R} .¹⁾ The following probabilities can be assigned to the five cases in (5):

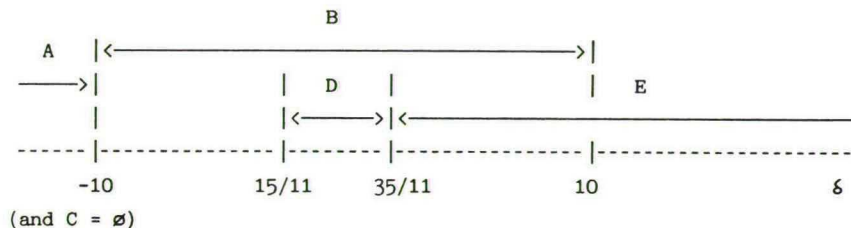
$$\begin{aligned}
 \Pr[A] &= \Pr[\delta \leq (-\beta w_1 - \gamma)/y_1] \\
 \Pr[B] &= \Pr[(-\beta w_1 - \gamma)/y_1 < \delta \leq (-\beta w_1 - \gamma + h_0)/y_1] \\
 \Pr[C] &= \Pr[(-\beta w_1 - \gamma + h_0)/y_1 < \delta \leq (-\beta w_2 - \gamma + h_0)/y_2] \\
 \Pr[D] &= \Pr[(-\beta w_2 - \gamma + h_0)/y_2 < \delta \leq (-\beta w_2 - \gamma + T)/y_2] \\
 \Pr[E] &= \Pr[(-\beta w_2 - \gamma + T)/y_2 < \delta]
 \end{aligned}$$

Let the parameter values be $\beta=20$ and $\gamma=0$, and let the budget constraint be characterized by $w_1=1$, $w_2=1/2$, $y_1=1$, $y_2=11$, $T=40$, so $h_0=20$ and $c_0=21$. Then we can identify the five cases with segments of the δ -axis as follows.



Thus, given the probability distribution of δ , the calculation of the probabilities is straightforward.

Now consider a second set of parameter values, $\beta=10$ and $\gamma=0$, with the same budget constraint as in the example above. The corresponding segments of the δ -axis are as follows.



Clearly, in this case the model is not coherent: It is not possible to solve h^* from (5) (as a function of δ). There is no reduced form corresponding to (5), so (5) is not a well-defined model.

Note that (1) has a unique solution with probability one, irrespective of whether (4) is satisfied or not. The root of the trouble is that (4) is not satisfied on the whole budget set; thus (5) may not be equivalent to (1) and may not have a unique solution.

It is easy to show that coherency of (5) is equivalent to

$$(h_0 - \beta w_1 - \gamma)/y_1 < (h_0 - \beta w_2 - \gamma)/y_2. \quad (6)$$

This condition appears to be equivalent to quasi-concavity of U at (h_0, c_0) for that value of δ for which (h_0, c_0) is the optimum, along both line segments. This can be shown straightforwardly as follows:

Assume that y_1 and y_2 are both positive and use the relation $y_1 + h_0 w_1 = y_2 + h_0 w_2 = c_0$. Then (6) can be simplified to

$$\beta y_2 > (h_0 - \beta w_2 - \gamma) h_0. \quad (7)$$

If, being on segment 2, the individual chooses h_0 , this implies $\delta = (h_0 - \beta w_2 - \gamma)/y_2$. Inserting this in (7) yields $\beta \delta h_0$, which is the concavity condition at (h_0, c_0) (with δ such that (h_0, c_0) is the optimum in case of the linear budget constraint characterized by w_2 and y_2). In the same way it is shown that (6) is equivalent to $\beta \delta h_0$, with δ such that (h_0, c_0) is the optimal point along segment 1.

In conclusion, concavity at the kink point for a specific value of δ is necessary and sufficient to avoid problems of incoherency. It is easy

to verify that concavity is satisfied at the kink point for the first set of parameter values but not for the second.

The model given by (1) is a simple example of a model with endogenous regimes due to a set of inequality constraints. The general framework is discussed in section 5. One of the goals of this paper is to discuss methods of avoiding coherency problems as encountered above. In this specific example there are two apparent ways to do this.

The first option is to restrict the range of possible realizations of the random variable δ and the value of the fixed parameter β . If δ is negative with probability one and β is non-negative, the problems do not arise, since in this case (4) is satisfied for all non-negative h .

Another possibility, which avoids truncation of the distribution of δ , is to impose (7) for all 'relevant' values of h_0 and c_0 . Notice that (7) can be rewritten as

$$\beta c_0 > (h_0 - \gamma)h_0 \quad (7')$$

Thus, if the fixed parameters β and γ are restricted such that (7') holds for all relevant (h_0, c_0) (e.g. all (h_0, c_0) in the sample), coherency is guaranteed. In a sense, the latter method is less restrictive than the first one, since it does not necessarily imply quasi-concavity of the direct utility function at all points of the budget set, not even at the optimum.²⁾

To conclude this example, a number of observations can be made. First of all, note that if the usual conditions for utility maximization (convex budget sets, convex preferences) hold, then endogenous variables are uniquely determined. This follows from standard Lagrange theory. Thus, 'regularity' implies coherency. Secondly, the reverse does not hold. An example is given in Figure 2. The unique point of tangency with the budget line satisfies first order conditions for utility maximization, but obviously does not represent a utility maximum. The econometric model, obtained by solving first order conditions, is coherent, but the micro-economic foundation of this model is lost.

In the third place, almost any specification used in practice will satisfy regularity conditions for utility maximization only locally. The main reason for this is the quest for flexible forms. Usually, flexibility is only possible if global concavity properties are sacrificed (cf., e.g., Diewert and Wales, 1987). In itself this is reasonable, as generally economic models only aim to describe behaviour of agents for a certain

range of exogenous variables. See for instance Figure 3, where the indifference curves are convex in a certain part of quantity space, but not everywhere. As long as attention is restricted to this 'regular area', no problems arise. Alternatively, and this is the approach in this paper, preference parameters can be restricted in such a way that indifference curves are convex in a given area of interest.

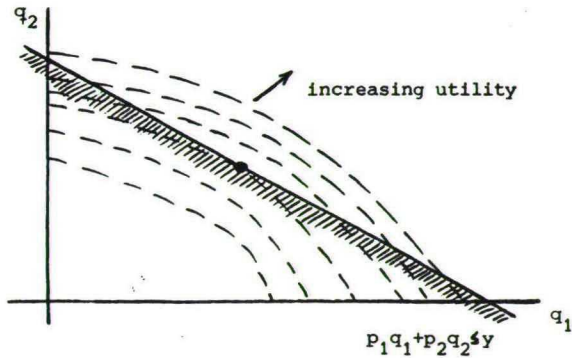


Figure 2. Coherency without regularity

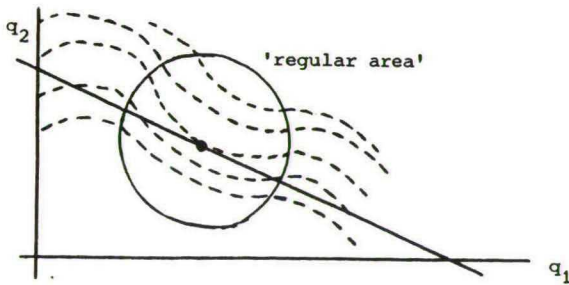


Figure 3. Local regularity of preferences

Example 2. Non-negativity constraints in the indirect translog demand system

Lee and Pitt (1986) consider the indirect translog demand system with binding non-negativity constraints:

$$s_i = \{\alpha_i + \sum_{j=1}^n \beta_{ij} \log v_j\} / D \quad (i=1, \dots, n),$$

where

$$D = -1 + \sum_{i=1}^n \sum_{j=1}^n \beta_{ij} \log v_j$$

β_{ij} : parameters ($i, j=1, \dots, n$)

n : number of goods

v_j : p_j/y with p_j the price of j -th good ($j=1, \dots, n$) and y income

s_i : budget share of good i

α_i : random parameters ($i=1, \dots, n$), representing random preferences,

$$\alpha_1 + \dots + \alpha_n = -1.$$

The case that the first l goods are not consumed and the other goods are consumed is characterized by the conditions

$$\begin{aligned} \pi_i(\bar{v}) &\leq v_i & (i=1, \dots, l), \\ x_i &> 0 & (i=l+1, \dots, n), \end{aligned}$$

where

$\pi_i(\bar{v})$: virtual price (or 'shadow price') of good i

\bar{v} : vector of market prices of the goods consumed in positive amounts

x_i : demand for good i given that the first l goods are not consumed.

The 'regime', i.e. the set of commodities for which non-negativity constraints are binding, is endogenously determined. Each regime is characterized by some subset of $\{1, \dots, n\}$, indicating which constraints are binding. For example, $\{1\}$ refers to the case for which the first commodity is the only one which is not consumed. Which regime occurs depends on the values of the α_i 's ($i=1, \dots, n$). Lee and Pitt (1986) show that each regime corresponds to some region in $(\alpha_1, \dots, \alpha_n)$ -space. Van Soest and Kooreman (1987) construct examples for $n=3$. Figure 4 gives one such example.

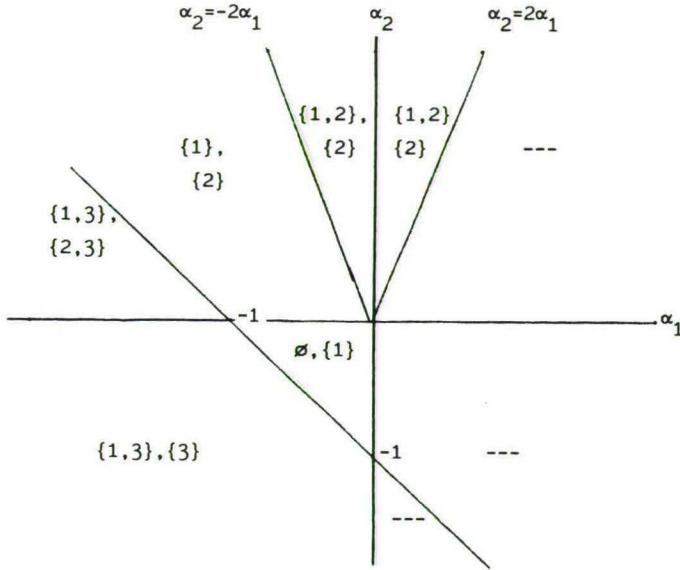


Figure 4. Incoherency in the indirect translog demand system with binding non-negativity constraints

$$\text{Let } B = [\beta_{ij}]_{i,j=1,\dots,3} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } v = (1,1,1)'$$

Figure 4 shows the regions in α -space ($\alpha = (\alpha_1, \alpha_2, -1 - \alpha_1 - \alpha_2)'$) corresponding to each of the 7 possible regimes. For α with $\alpha_1 > 0$ and $\alpha_2 < 2\alpha_1$, no solution is found, and for other α 's (except for some set of probability zero) there are two corresponding regimes, implying that different regimes occur simultaneously. This implies that, given α , the regime is not uniquely determined. As a consequence, the vector of endogenous variables $(q_1, \dots, q_n)'$ cannot be solved from the structural model. No reduced form exists, or, in other words, the model is incoherent.

For other parameter values, i.e. different values of β_{ij} , such problems need not arise. Van Soest and Kooreman (1987) give sufficient conditions to avoid the incoherency. It turns out that these same conditions also guarantee concavity of the cost function in some relevant

region of quantity space. Thus, a strong connection is suggested between concavity of the cost function and coherency of the demand system. \square

3. Incoherency and ML-estimation

Although the requirement that a model should be coherent may appear self-evident, one may still ask whether imposition of coherency conditions is strictly necessary. After all, given that the true data generating process is coherent, one might hope that parameter estimates automatically converge to values which satisfy coherency conditions. Also, one may ask whether it is possible to test coherency conditions imposed on parameters. This section addresses these issues by means of yet another example.

We consider the following simultaneous Probit-model (see e.g. Schmidt, 1981).

$$\begin{aligned} y_1^* &= \beta_1 x + \gamma_1 y_2 + \varepsilon_1 \\ y_2^* &= \beta_2 x + \gamma_2 y_1 + \varepsilon_2 \end{aligned} \quad (8)$$

$$y_i = 1 \text{ if } y_i^* > 0 \text{ and } y_i = 0 \text{ if } y_i^* \leq 0 \quad (i=1,2)$$

Here x denotes an (observable) exogenous variable, y_1^* and y_2^* are latent endogenous variables, y_1 and y_2 are observed endogenous variables and ε_1 and ε_2 are random variables following a bivariate normal distribution:

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \sim N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

The vector of parameters to be estimated is $\theta = (\beta_1, \beta_2, \gamma_1, \gamma_2) \in \Theta^* = \mathbb{R}^4$.

It is straightforward to derive the probabilities of the four different outcomes which are possible:

$$\begin{aligned} \Pr[y_1=0, y_2=0] &= \Phi(-\beta_1 x) \Phi(-\beta_2 x), \\ \Pr[y_1=0, y_2=1] &= \Phi(-\beta_1 x - \gamma_1) \Phi(\beta_2 x), \\ \Pr[y_1=1, y_2=0] &= \Phi(\beta_1 x) \Phi(-\beta_2 x - \gamma_2), \text{ and} \\ \Pr[y_1=1, y_2=1] &= \Phi(\beta_1 x + \gamma_1) \Phi(\beta_2 x + \gamma_2), \end{aligned} \quad (9)$$

where Φ denotes the standard normal cumulative density function.

If the model is not coherent, then the four probabilities in (9) do not sum to one. In general, their sum equals

$$1 + [\Phi(\beta_1 x + \gamma_1) - \Phi(\beta_1 x)] [\Phi(\beta_2 x + \gamma_2) - \Phi(\beta_2 x)] \quad (10)$$

so that a condition for coherency is that $\gamma_1 \gamma_2 = 0$. This renders the model recursive (cf. Schmidt, 1981). The condition given here is a special case of the coherency conditions given by Gourieroux et al. (1980).

Let $\Theta = \{(\beta_1, \beta_2, \gamma_1, \gamma_2); \gamma_1 \gamma_2 = 0\}$. If the model is coherent, i.e. $\theta \in \Theta$, then (9) implies that the log-likelihood function of a random sample $(y_1, x_1), \dots, (y_N, x_N)$ can be written as

$$\begin{aligned} L(\theta) = & \sum_{t \in I_{00}} \log[\Phi(-\beta_1 x_t) \Phi(-\beta_2 x_t)] + \sum_{t \in I_{01}} \log[\Phi(-\beta_1 x_t - \gamma_1) \Phi(\beta_2 x_t)] + \\ & + \sum_{t \in I_{10}} \log[\Phi(\beta_1 x_t) \Phi(-\beta_2 x_t - \gamma_2)] + \sum_{t \in I_{11}} \log[\Phi(\beta_1 x_t + \gamma_1) \Phi(\beta_2 x_t + \gamma_2)]. \end{aligned} \quad (11)$$

Here t denotes the observation and $I_{i,j} = \{t; y_{1t} = i \text{ and } y_{2t} = j\}$. If $\theta \notin \Theta$, the expressions in (9) can still be computed, although their interpretation is not clear. Thus, (11) can also be computed for $\theta \in \Theta^* \setminus \Theta$. This defines a natural extension of the likelihood function from Θ to Θ^* .

Let us now assume that the true parameters of the data generating process are

$$\beta_1 = 1, \gamma_1 = -1, \beta_2 = 0, \text{ and } \gamma_2 = 0 \quad (12)$$

Note that $\gamma_2 = 0$ implies that the true model is coherent. Furthermore, we assume that the exogenous variable x is a dummy variable with value 1 for half of the observations and value 0 for the other half. Inserting the true parameter values in (9) yields (subscript t is omitted)

$$\begin{aligned} \Pr[y_1 = 0, y_2 = 0 | x = 0] &= 0.250; \Pr[y_1 = 0, y_2 = 0 | x = 1] = 0.079, \\ \Pr[y_1 = 0, y_2 = 1 | x = 0] &= 0.421; \Pr[y_1 = 0, y_2 = 1 | x = 1] = 0.250, \\ \Pr[y_1 = 1, y_2 = 0 | x = 0] &= 0.250; \Pr[y_1 = 1, y_2 = 0 | x = 1] = 0.421, \\ \Pr[y_1 = 1, y_2 = 1 | x = 0] &= 0.079; \Pr[y_1 = 1, y_2 = 1 | x = 1] = 0.250. \end{aligned} \quad (13)$$

For a random sample of $2T$ observations, T with $x=0$ and T with $x=1$, let $K(i, j, k)$ be the number of observations with $y_1 = i$, $y_2 = j$, and $x = k$ ($i, j, k \in \{0, 1\}$). Note that

$$\lim_{T \rightarrow \infty} \{K(i, j, k)/T\} = \Pr[y_1 = i, y_2 = j | x = k] \quad (i, j, k \in \{0, 1\}).$$

If the model is not coherent, the likelihood function is not defined. Thus, strictly speaking, maximum likelihood estimation requires a priori imposition of coherency conditions. Still, there are many examples, including the example at hand, in which the likelihood function has a natural extension to the set of parameters for which the model is not coherent. In this example, the extension is given by (11). The researcher who does not impose coherency a priori will consider (11) as the likelihood function for all $\theta \in \Theta^*$ and estimate the parameters by maximizing (11) over Θ^* . Our purpose is to show that the resulting estimator for the parameter β_2 is inconsistent.

The extended log-likelihood (11) can be rewritten as

$$L(\beta_1, \beta_2, \gamma_1, \gamma_2) = L_1(\beta_1, \gamma_1) + L_2(\beta_2, \gamma_2).$$

where

$$L_1(\beta_1, \gamma_1) = \sum_{j,k=0}^1 K(0,j,k) \log \Phi(-\beta_1 k - \gamma_1 j) + \sum_{j,k=0}^1 K(1,j,k) \log \Phi(\beta_1 k + \gamma_1 j)$$

and

$$L_2(\beta_2, \gamma_2) = \sum_{i,k=0}^1 K(i,0,k) \log \Phi(-\beta_2 k - \gamma_2 i) + \sum_{i,k=0}^1 K(i,1,k) \log \Phi(\beta_2 k + \gamma_2 i).$$

Maximization of L is thus achieved by maximizing L_1 (with respect to β_1 and γ_1) and L_2 (with respect to β_2 and γ_2) separately. This means that the two simultaneous Probit-equations are treated as if they were separate Probit-equations, i.e. maximizing L over Θ^* yields a consistent estimator if y_2 is exogenous in the first equation and y_1 is exogenous in the second equation. Since the true value of γ_2 is 0, y_2 is independent from ϵ_1 and there is nothing wrong in estimating the parameters of the first equation in this way, i.e. the estimators for β_1 and γ_1 are consistent. The estimators for β_2 and γ_2 however are inconsistent, as can be shown by a straightforward computation of their probability limits: In the limiting case ($T \rightarrow \infty$), the sample fractions $K(i,j,k)/T$ equal the probabilities given by (13) and we can write

$$\begin{aligned} \text{plim}_{T \rightarrow \infty} \{L_2(\beta_2, \gamma_2)/T\} = & 0.250 \log \Phi(0) + 0.079 \log \Phi(-\beta_2) + 0.250 \log \Phi(-\gamma_2) + \\ & 0.421 \log \Phi(-\beta_2 - \gamma_2) + 0.421 \log \Phi(0) + 0.250 \log \Phi(\beta_2) + \\ & 0.079 \log \Phi(\gamma_2) + 0.250 \log \Phi(\beta_2 + \gamma_2). \end{aligned}$$

Since L_2 has the form of the log-likelihood of a Probit model, it is globally concave. Its unique maximum can easily be found numerically. In the limiting case ($T \rightarrow \infty$), it is attained for $\beta_2 = 0.5726$ and $\gamma_2 = -0.8405$. Thus we have

$$\text{plim}_{T \rightarrow \infty} \hat{\beta}_2 = 0.5726 \neq 0 = \beta_2 \text{ and } \text{plim}_{T \rightarrow \infty} \hat{\gamma}_2 = -0.8405 \neq 0 = \gamma_2.$$

Finally, note that if the restriction $\gamma_1 \gamma_2 = 0$ is imposed (which is necessary and sufficient for coherency), then the resulting estimator for β_2 is consistent.³⁾

This example shows that 'Maximum Likelihood' estimation is not appropriate if coherency is not guaranteed for all values in the parameter space on which the likelihood function is to be maximized. Even if the model is coherent for the true parameter values, the ML-technique may yield inconsistent estimators and can lead to the conclusion that the model is not coherent: For a large enough sample, the null hypothesis $\gamma_1 \gamma_2 = 0$ would be rejected using standard methods of statistical inference. Moreover, the example shows that even the estimates of parameters which have no direct relation to the coherency condition (β_2 in the example) can be inconsistent. It thus makes clear that coherency is a *conditio sine qua non* for the use of Maximum Likelihood.

One can argue that the example of the bivariate probit model is not appropriate, since in this example coherency is well-known to be an issue and coherency conditions are easy to impose. Thus the fear that the empirical researcher will fail to impose coherency conditions might be unjustified for this specific example. In many, more complicated, models however, it is much less clear that coherency might be a problem and it is also less clear how coherency conditions should be imposed. In such cases the researcher will be tempted to start with the estimation without paying attention to coherency. The example above demonstrates that such an approach may lead to an inconsistent estimator and incorrect procedures of statistical inference.

4. Demand Systems with Fixed Regimes

In this section we describe a general framework for the estimation of a standard demand system, an inverse demand system, and a conditional demand system. We impose consistency with utility maximizing behaviour and conditions which guarantee that the dual approach is appropriate, i.e. Shephard's Lemma and Roy's Identity can be applied in order to derive the standard demand functions, starting from the expenditure function or the indirect utility function, respectively. We first introduce some notation and standard regularity conditions. Next we consider restrictions on the parameter space that can be imposed in estimation to ensure that the estimated system satisfies the regularity conditions. Finally, another condition, called 'external coherency', is introduced, which states that the model must be able to explain enough features of the data.

Regularity conditions

We assume that each individual maximizes some direct utility function subject to a linear budget constraint. Topics such as rationing, non-negativity constraints, and kinked budget constraints are discussed below. We start from an indirect utility function v_θ given by

$$u = v_\theta(p, y) \quad ((p, y) \in \tilde{V}_\theta \subset \mathbb{R}^n \times \mathbb{R}),$$

where $p = (p_1, \dots, p_n)'$ is a vector of prices of n commodities,
 y denotes income (or total expenditures on the n commodities),
 u is the utility level, and
 $\theta \in \Theta \subset \mathbb{R}^m$ is a vector of (fixed or random) parameters.

Standard regularity conditions for given $\theta \in \Theta$ are (see, e.g., Barten and Böhm, 1982):

- A1. v_θ is homogeneous of degree 0:
for all $(p, y) \in \tilde{V}_\theta$ and $\lambda \in \mathbb{R}^+$, $(\lambda p, \lambda y) \in \tilde{V}_\theta$ and $v_\theta(\lambda p, \lambda y) = v_\theta(p, y)$.
- A2. v_θ is twice continuously differentiable with respect to prices and income and for all $(p, y) \in \tilde{V}_\theta$, $(\partial v_\theta / \partial y)(p, y) > 0$.

Assumption A2 implies that v_θ is strictly increasing in y and allows for the introduction of the expenditure or cost function e_θ on the set

$\tilde{E}_\theta = \{(p, v_\theta(p, y)); (p, y) \in \tilde{V}_\theta\}$. e_θ is implicitly defined by

$$v_\theta(p, e_\theta(p, u)) = u \quad ((p, u) \in \tilde{E}_\theta).$$

The dual approach is only consistent with utility maximizing behaviour if 'strict' concavity is guaranteed. More precisely: e_θ is said to be regular at given $(p, u) \in \tilde{E}_\theta$ if the $n \times n$ matrix $(\partial^2 e_\theta / \partial p \partial p')(p, u)$ is negative semi-definite and of rank $n-1$. v_θ is said to be regular at $(p, y) \in \tilde{V}_\theta$ if e_θ is regular at $(p, v_\theta(p, y))$. With these definitions the third regularity condition can be formulated:

A3. v_θ is regular at all $(p, y) \in \tilde{V}_\theta$.

In what follows we work with a convex subset V_θ of \tilde{V}_θ , and we assume that A1-A3 are satisfied on V_θ . V_θ is referred to as the regular set in (p, y) -space. Marshallian (uncompensated) demand functions are denoted by

$$q = F_\theta(p, y) \quad ((p, y) \in V_\theta),$$

where $q = (q_1, \dots, q_n)'$ is a vector of (not necessarily non-negative) quantities and the components of the vector-valued function F_θ , defined on V_θ , are, according to Roy's identity, given by

$$F_{\theta, i}(p, y) = -(\partial v_\theta / \partial p_i)(p, y) / (\partial v_\theta / \partial y)(p, y) \quad (i=1, \dots, n).$$

The regular set in q -space, $Q_\theta \subset \mathbb{R}^n$, is defined as

$$Q_\theta = \{F_\theta(p, y); (p, y) \in V_\theta\}.$$

The assumptions A1-A3 together with the convexity of V_θ imply that F_θ is homogeneous of degree zero and one-to-one from $\{(p, 1) \in V_\theta\}$ onto Q_θ (see Gale and Nikaido, 1965).

Parameterization and restrictions in the parameter space

Preference variation across individuals (or households) can be incorporated in the parameter vector θ . For each individual t , we write

$$\theta_t = g_t(\psi, \eta_t).$$

Here ψ is a vector (or matrix) of fixed parameters (with the same value for all individuals) chosen from some set Ψ , and the vectors η_t are independent drawings from some probability distribution which does not depend on t . The (vector-valued) function g_t may depend on t , e.g. through a vector x_t of observed individual characteristics. The most common example is $g_t(\psi, \eta_t) = \psi x_t + \eta_t$, where ψ is a matrix of appropriate size. Thus, systematic preference variation is allowed for if g_t depends on t , whereas the presence of the η_t 's implies random variation of preferences.

In estimating the system of demand equations, the following conditions may be imposed on the set Ψ of possible values of ψ and/or on the support Ω of the distribution of the η_t 's (i.e., loosely speaking, the set of all possible realizations of the η_t 's). The conditions are imposed to guarantee regularity of preferences in all relevant points of price and/or quantity space. Here 'relevant points' include observed points, but may also include e.g. points for which model simulations are performed. Which condition is appropriate depends on the type of model to be estimated. In each case, the conditions are sufficient but in general not necessary for coherency.

Condition B1 is appropriate in case the model to be estimated is a standard demand system, i.e. Marshallian demand functions are estimated and the linear budget constraint is the only binding constraint in the model.

B1. (Regularity in a minimal subset of (p, y) -space)

$$\text{For all } t, \text{ for all } \psi \in \Psi, \text{ and for all } \eta \in \Omega: V_{g_t(\psi, \eta)} \supset V_{\min}.$$

This condition states that for all parameter values (and thus for all possible individual preference structures) the model must be able to explain behaviour for at least some minimal subset, V_{\min} , of (p, y) -space. This subset must contain all observed (p, y) 's in the sample, in order to guarantee that the demand system is defined and regular at each data-point.⁴⁾ If the model is used for simulations with values of (p, y) outside the range of (p, y) 's in the sample, then V_{\min} must contain these extra (p, y) values also. B1 implies that the parameter space Θ cannot be too large; otherwise there might be values of ψ or η_t such that at some points of V_{\min} conditions A1-A3 are not satisfied. V_{\min} can e.g. be rectangular: $V_{\min} = \{(p, y) \in \mathbb{R}^n \times \mathbb{R}; (p, \underline{y}) \leq (p, y) \leq (p, \bar{y})\}$, for given $p, \underline{p}, \bar{p}, \underline{y}$.

Condition B1 is illustrated in Figure 5. Here a chosen set V_{\min} in (p,y) -space and (for some given ψ and t) the regular area's $V_{g_t}(\psi, \eta)$ are given for two different values η_1 and η_2 of η . If preferences are characterized by η_1 , behaviour cannot be explained if, e.g., $(p,y)=(p_0,y_0) \in V_{\min}$. Therefore η_1 must be excluded from Ω . For $\eta=\eta_2$, the model can explain behaviour for all $(p,y) \in V_{\min}$, so η_2 may be included in Ω . Thus for given ψ and t condition B1 implies a restriction on Ω . Together these restrictions imply that Ω and Ψ cannot be too large.

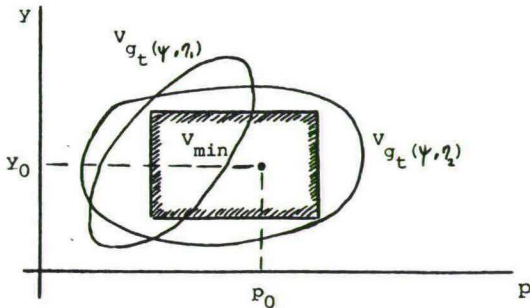


Figure 5. Condition B1 in (p,y) -space

Condition B2 is the counterpart of B1 in quantity space. It can be used, for example, in case an inverse demand system, i.e. inverse Marshallian demand functions, is to be estimated. See, e.g., Anderson (1980), for some theoretical properties, and Barten and Bettendorf (1989) for an empirical application. In empirical practice, there are not many situations in which estimation of an inverse demand system is relevant. Condition B2 however also appears to be important in the case of endogenously switching regimes due to inequality constraints. See section 6. The condition states that, for given (fixed and random) parameter values, certain quantity vectors must be optimal for some prices and income. As with B1 this implies that the parameter space cannot be too large.

B2. (Regularity in a minimal subset of q -space)

For all t , for all $\psi \in \Psi$, and for all $\eta \in \Omega$: $Q_{g_t}(\psi, \eta) \supset Q_{\min}$.

Condition B2 is illustrated in Figure 6. For given parameter values $g_t(\psi, \eta)$, the quantity space consists of three parts: The area where the direct utility function is not defined (because shadow prices do not exist) (QN), the area where indifference curves exist but are not convex (QI), and the regular area $Q_{g_t(\psi, \eta)}$. The condition states that parameters have to be restricted such that Q_{\min} is contained in $Q_{g_t(\psi, \eta)}$.

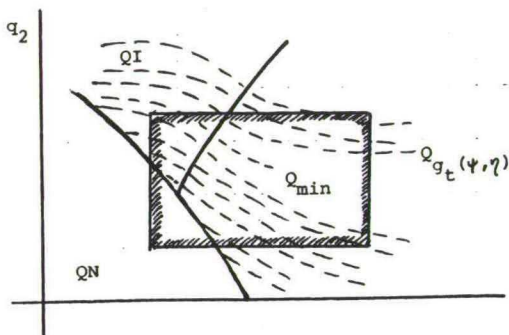


Figure 6. Condition B2 in q-space

Conditions B1 and B2 are similar in the sense that they both define an area in q-space where indifference curves must be convex. However, since restrictions are imposed a priori, i.e. before parameters are estimated, it is not possible to tell which point in q-space corresponds to a particular (p,y)-combination. Thus, if we choose a particular V_{\min} and estimate the parameters imposing B1, it may turn out that indifference curves are convex in an area quite different from the Q_{\min} we had in mind. Similarly, a choice of Q_{\min} and imposition of B2 may actually imply concavity on an area in (p,y)-space quite different from V_{\min} . This point is illustrated for the special case of the quadratic utility function in section 5.

Before we introduce a condition similar to B1 and B2 which is useful if a conditional demand system must be estimated, we first present a simple example to show why an explicit condition for this case is necessary. The example shows that it is not sufficient to impose conditions B1 and/or B2, since it is necessary to take explicit account of the way in which (p,y)-space and q-space are related.

An Example with Rationing

Consider the following Gorman Polar Form expenditure function for three goods, defined for $p_i > 0$ and $a_i > 0$ ($i=1,2,3$).

$$e(u, p_1, p_2, p_3) = -1/2(p_2^2/p_3) \exp(p_1/p_3) - p_3 \exp(p_2/p_3) + \sum_{i=1}^3 a_i p_i + up_3 \quad (14)$$

The 2x2 submatrix of second order derivatives with respect to p_1 and p_2 is

$$- \frac{1}{p_3} \begin{bmatrix} 1/2 v_2^2 \exp(v_1) & v_2 \exp(v_1) \\ v_2 \exp(v_1) & \exp(v_1) + \exp(v_2) \end{bmatrix}$$

where $v_1 = p_1/p_3$, $v_2 = p_2/p_3$. This matrix is negative definite for $v_1 < v_2$. The demand functions for goods 1 and 2 are

$$q_1 = -1/2 v_2^2 \exp(v_1) + a_1 \quad (15a)$$

$$q_2 = -v_2 \exp(v_1) - \exp(v_2) + a_2 \quad (15b)$$

Note that demand for goods 1 and 2 does not depend on income. Suppose now that q_1 is rationed at $q_1 = \bar{q}_1$. We know from rationing theory (c.f., e.g., Deaton and Muellbauer, 1980), that q_2 is then obtained by first solving v_1 from (15a), for given v_2 and $q_1 = \bar{q}_1$, and then inserting the solution (\tilde{v}_1 , say) in (15b). Let us assume that $\bar{q}_1 = -1 + a_1$. This is a perfectly feasible value; it is generated by, for instance, $(v_1, v_2) = (\log 2, 1)$. For this price vector negativity is satisfied, since $v_1 < v_2$.

Now assume however that $\bar{q}_1 = -1 + a_1$ and $v_2 = 1/2$. Then $\tilde{v}_1 = \log 8 > v_2$. Hence, there is no shadow price \tilde{v}_1 for which the expenditure function is concave, even though \bar{q}_1 and v_2 are both feasible. It is the combination $\bar{q}_1 = -1 + a_1$ and $v_2 = 1/2$ which causes problems.

This example shows that the relationship between negativity and existence of a feasible solution of the rationing problem is not straightforward. It is not sufficient to know the regions in price space and in quantity space where concavity holds. In case of rationing, part of the price vector and part of the quantity vector are given. Conditional and inverse conditional demand functions are necessary to determine the quantity vector and the price vector to which the given mixed price and quantity vector corresponds, and are therefore necessary to determine whether the given mixed vector is feasible. □

Since the example above shows that regularity in a conditional demand system cannot be stated in terms of B1 and B2, we introduce another condition, B3, for this case. First, we need some extra notation.

A conditional demand system is the solution of the problem of utility maximization, not only subject to the budget constraint, but also subject to an a priori given set of equality constraints on a number of quantities. We assume that the quantities of goods 1 through k are constrained, whereas the other quantities can be chosen freely. The number k and the order of the goods may vary across individuals, i.e. different individuals may face different constraints.

The individual thus solves the problem

$$\max_{q_{II}} U_{\theta}(\bar{q}_I, q_{II}) \quad \text{s.t.} \quad y = p_I' \bar{q}_I + p_{II}' q_{II}.$$

Here we have written $q = (q_I', q_{II}')'$, $p = (p_I', p_{II}')'$, and the constraints are given by $q_I = \bar{q}_I$, where $q_I = (q_1, \dots, q_k)'$, etc.

Starting from the indirect utility function v_{θ} and corresponding demand system F_{θ} as introduced above, the solution of this maximization problem can be found using shadow (or virtual) prices (see, e.g., Neary and Roberts, 1980):

Find $\tilde{p}_I \in \mathbb{R}^k$, $\tilde{y} \in \mathbb{R}$ and $q_{II} \in \mathbb{R}^{n-k}$ such that

$$\left\{ \begin{array}{l} ((\tilde{p}_I, p_{II}), \tilde{y}) \in v_{\theta} \\ F_{\theta}((\tilde{p}_I, p_{II}), \tilde{y}) = (\bar{q}_I, q_{II}) \\ \tilde{y} = y + (\tilde{p}_I - p_I)' \bar{q}_I \end{array} \right. \quad (16)$$

The optimal quantities, taking into account the constraints $q_I = \bar{q}_I$, are then given by q_{II} .

We are now ready to formulate condition B3:

B3. Let, for each t, VQ_t^I be a given subset of $\{(p, y, q_I) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k\}$. Then for all $\psi \in \Psi$, for all $\eta \in Q$, for each t, and for all $(p, y, q_I) \in VQ_t^I$, there must exist $\tilde{p}_I \in \mathbb{R}^k$, $\tilde{y} \in \mathbb{R}$ and $q_{II} \in \mathbb{R}^{n-k}$ such that

$$\left\{ \begin{array}{l} ((\tilde{p}_I, p_{II}), \tilde{y}) \in V_{g_t}(\psi, \eta) \\ F_{g_t}(\psi, \eta)((\tilde{p}_I, p_{II}), \tilde{y}) = (q_I, q_{II}) \\ \tilde{y} = y + (\tilde{p}_I - p_I)' q_I \end{array} \right.$$

This condition states that, for each parameter vector, (16) must have a regular solution. It is necessary to guarantee that the domain of the conditional demand functions is large enough and thus sufficient for coherency of the conditional demand system: It states that to each $\psi \in \Psi$ and each $\eta \in \Omega$, there corresponds at least one vector q_{II} of endogenous variables, and together with concavity of the expenditure function and convexity of V_g this implies coherency.

If there are no measurement errors on prices, income or rationed quantities, then VQ_t^I must at least contain the observed (p_t, y_t, \bar{q}_{It}) .

Conditions B1, B2, and B3 are similar in the sense that they all impose restrictions in a subspace of (p, y, q) -space which must hold for all feasible parameter values. All three conditions imply that the parameter space cannot be too large. Which conditions are useful not only depends on the model to be estimated, but also on what the model is used for. For example, if a conditional demand system is used for simulations in which quantity constraints are relaxed, then both B3 and B1 must be imposed.

Conditions B1-B3 imply that the space of fixed and random parameters cannot be too large and thus limit the flexibility of the error structure of the model. Particularly if optimization and measurement errors are excluded, the specification of the model must ensure that the model can explain observed behaviour. In other words, all likelihood contributions must be non-zero. This implies that the parameter space cannot be too small. For the case of a standard or inverse demand system, this condition can be formulated as follows.

B4. ('External coherency')

Let, for all t , VQ_t be a given subset of $\{(p, y, q) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n; p'q = y\}$.

Then for all t , for all $\psi \in \Psi$, and for all $(p, y, q) \in VQ_t$, there must be an $\eta \in \Omega$ such that $F_{g_t}(\psi, \eta)(p, y) = q$.

One can think of VQ_t as the set of prices, incomes and quantities which may arise for observation t . If no measurement or optimization errors are

involved, VQ_t must at least contain the observed (p,y,q) -vector for individual t . In fact, VQ_t may then consist of just one element. The condition states that random preferences η must guarantee so much flexibility that for all $\psi \in \Psi$ and at least one possible value of η a given (observed) quantity vector is optimal for given prices and income. This motivates the term 'external coherency': The model has to be coherent with available data, in the sense that the likelihood contribution of any given data point must be strictly positive. Rather than by imposing this condition, this may also be achieved by explicit incorporation of measurement or optimization errors in the model.

Figure 7 gives an example for one data point (i.e. a particular budget line and a point A on it). For each value of ψ there must be at least one value of $\eta \in \Omega$ which produces an indifference curve tangent to the budget line at A.

Note the conflicting nature of B1 and B2 on the one hand and B4 on the other hand. B1 and B2 may for instance imply such strong restrictions on Ω that for a given observed (p,y) (and for given t and ψ), demand $F_{g_t}(\psi, \eta)$ equals the observed q for no $\eta \in \Omega$, so that B4 is not satisfied. This would be a consequence of the fact that B1 and B2 imply that, for given Ψ , Ω cannot be too large, whereas B4 implies that it cannot be too small.

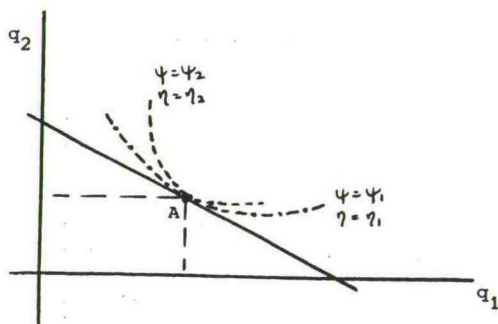


Figure 7. Condition B4 in q-space

For the case of a conditional demand system, an 'external coherency' condition quite similar to B4 can be formulated:

B5. ('External coherency')

Let VQ_t be a given subset of $\{(p, y, q) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n; p'q = y\}$. Then for all $\psi \in \Psi$ and all $(p, y, q) \in VQ_t$, there must be at least one $\eta \in Q$ such that there are $\tilde{p}_I \in \mathbb{R}^k$ and $\tilde{y} \in \mathbb{R}$ with

$$\left\{ \begin{array}{l} ((\tilde{p}_I, p_{II}), \tilde{y}) \in V_{g_t}(\psi, \eta) \\ F_{g_t}(\psi, \eta)((\tilde{p}_I, p_{II}), \tilde{y}) = q \\ \tilde{y} = y + (\tilde{p}_I - p_I)'q_I \end{array} \right.$$

This condition states that certain quantity vectors q_{II} can be optimal for given prices, income and rationed quantities q_I . If the model contains no optimization or measurement errors, VQ_t must at least contain the observed vector $(p_t, y_t, (\bar{q}_{It}, q_{IIIt}))$. It guarantees, similar to B4, that each data point has a non-zero likelihood contribution. Note however that B5 is weaker than B4 since quantities \bar{q}_{It} do not have to be rationalized.

The reader must be well aware of the difference between B5 and B3: B3 states that for each $\eta \in Q$ the rationed utility maximization problem has some regular solution. B5 states that the model must be able to explain a given observation, i.e., to each observed optimum there must correspond some $\eta \in Q$.

5. Examples

In this section, a number of examples are given to illustrate the implications of the conditions introduced in the previous section and to show how the conditions can be imposed in practice.

Example 1: Linear Expenditure System (LES)

The indirect utility function is defined for positive prices:

$$v_\theta(p, y) = (y - p'y) \prod_{k=1}^n p_k^{-\alpha_k}, \quad (17)$$

with $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i > 0$ ($i=1, \dots, n$), $p = (p_1, \dots, p_n)'$, $y = (y_1, \dots, y_n)'$.

The expenditure function is

$$e_{\alpha, \gamma}(p, u) = p' \gamma + u \prod_{k=1}^n p_k^{\alpha_k}.$$

A typical element of the matrix of second order derivatives is

$$\{(\partial^2 e_{\alpha, \gamma} / \partial p \partial p')(p, u)\}_{i, j} = \frac{u}{p_i p_j} \left\{ \prod_{k=1}^n p_k^{\alpha_k} \right\} \{\alpha_i \alpha_j - \delta_{ij} \alpha_i\},$$

with $\delta_{ij}=1$ if $i=j$, $\delta_{ij}=0$ if $i \neq j$.

Given the assumptions on the α_i , this matrix is negative semi-definite if and only if $u > 0$ (cf., e.g., Lemma 1 in Van Soest and Kooreman, 1987). In view of (17), u is positive if and only if $y - p' \gamma > 0$ so that a maximal choice for the regular set in (p, y) -space is given by

$$V_{\alpha, \gamma} = \{(p, y); p > 0 \text{ and } y - p' \gamma > 0\}.$$

The Marshallian demand functions are

$$q_i = F_{\theta, i}(p, y) = \gamma_i + (\alpha_i / p_i)(y - p' \gamma) \quad (i=1, \dots, n). \quad (18)$$

From the definition of Q_{θ} it follows that the corresponding regular region in q -space is given by

$$Q_{\alpha, \gamma} = \{q \in \mathbb{R}^n; q > \gamma\}.$$

Random preferences can be incorporated as follows:

$$\gamma_t = \gamma_{t,0} + \eta_t,$$

where $\gamma_{t,0} = (\gamma_{t1,0}, \dots, \gamma_{tn,0})'$ is fixed (and may depend on personal characteristics of individual t) and $\eta_t = (\eta_{t1}, \dots, \eta_{tn})'$ is random (with a distribution that does not depend on t).

We first elaborate conditions B1, B2 and B4.

B1. The random variable η_t should be restricted such that $y - \gamma_t' p > 0$ for all $(p, y) \in V_{\min}$. If, for instance, we take V_{\min} to be the rectangle

$$V_{\min} = \{(p, y); 0 \leq p \leq \bar{p}, 0 \leq y \leq \bar{y}\},$$

then the η 's have to satisfy

$$\max_{0 \leq p \leq \bar{p}} p' \eta < -\max_t \max_{0 \leq p \leq \bar{p}} p' \gamma_{t,0}, \quad (19)$$

which then defines the largest possible Ω for given $\gamma_{t,0}$'s.

B2. Let $Q_{\min} = \{q; q > \bar{q}\}$, with \bar{q} some given vector. B2 states that the γ 's have to satisfy $\gamma < \bar{q}$, or, equivalently, $\eta < \bar{q} - \gamma_{t,0}$ for all t .

B4. Solving γ from (18) yields

$$\gamma_i = q_i - \lambda \alpha_i / p_i \quad (i=1, \dots, n), \text{ for some arbitrary } \lambda > 0.$$

If we define VQ_t as $\{(p_t, y_t, q_t)\}$ (with $p_t' q_t = y_t$), then Ω has to be so large that for each t it contains at least one value of η in the set

$$\{\eta \in \mathbb{R}^n; \eta_i = q_{ti} - \lambda \alpha_i / p_{ti} - \gamma_{ti,0} \quad (i=1, \dots, n) \text{ for some } \lambda > 0\}.$$

Note that by choosing λ large enough we can always guarantee that η will be in Ω according to (19). Hence, conditions B1 and B4 can be satisfied simultaneously.

It may be illuminating to discuss the role of B1, B2, and B4 for LES a bit further. Suppose we do not allow for measurement errors. Then B1 and B2 imply restrictions on the range of the random variables η which depend on parameters implicit in $\gamma_{t,0}$. This in itself may give rise to non-standard estimation problems. The imposition of B1 and B2 makes sure that whatever the estimates will be, the resulting model will be consistent with neoclassical theory. Moreover, B4 implies that the likelihood is non-zero for all data points.

In order to be able to analyse B3 and B5 for LES, we first derive the conditional demand system. Solving

$$\max_{q_{II}} U_{\theta}(\bar{q}_I, q_{II}) = \prod_{i=1}^k (\bar{q}_i - \gamma_i)^{\alpha_i} \prod_{i=k+1}^n (q_i - \gamma_i)^{\alpha_i} \quad \text{s.t. } y = p_I' \bar{q}_I + p_{II}' q_{II}$$

yields

$$q_i = \gamma_i + \alpha_i (y - p_I' \bar{q}_I - p_{II}' \gamma_{II}) / \{p_i \sum_{j=k+1}^n \alpha_j\} \quad (i=k+1, \dots, n), \quad (20)$$

where $\gamma_{II} = (\gamma_{k+1}, \dots, \gamma_n)'$.

Alternatively, (20) can be obtained by first solving the shadow prices $(\tilde{p}_1, \dots, \tilde{p}_k)'$ from the first k unconditional demand equations, with p_I replaced by \tilde{p}_I and y replaced by \tilde{y} , yielding

$$\tilde{p}_i = \alpha_i (y - p_1' \bar{q}_1 - p_2' \gamma_2) / \{ (\bar{q}_i - \gamma_i) \sum_{j=k+1}^n \alpha_j \} \quad (i=1, \dots, k),$$

(21)

and

$$\tilde{y} = y + \sum_{j=1}^k (\tilde{p}_j - p_j)' \bar{q}_j,$$

and next substituting the solution into the notional demand equations for goods $k+1$ through n , again with p_I replaced by \tilde{p}_I and y replaced by \tilde{y} . The solution is feasible iff $\bar{q}_I > \gamma_I$ and $q_2 > \gamma_{II}$, or, equivalently, $\tilde{p}_I > 0$, $p_{II} > 0$, and $y - \tilde{p}_I' \gamma_I - p_{II}' \gamma_{II} > 0$.

Note that equation (20) has the same functional form as the notional demand functions (18), the only difference being that α_i is replaced by $\alpha_i / \{\alpha_{k+1} + \dots + \alpha_n\}$ and y is replaced by $y - p_I' \bar{q}_I$, and that (20) does not depend on $(\alpha_1, \dots, \alpha_k)$ nor on $(\gamma_1, \dots, \gamma_k)$.

Conditions B3 and B5 can be elaborated as follows:

B3. Let $VQ_t^I = \{(p_t, y_t, q_{It})\}$. Existence of a feasible solution for given $\gamma = (\gamma_I, \gamma_{II})$ means $q_{It} > \gamma_I$ and $y - p_{It}' q_{It} - p_{II}' \gamma_{II} > 0$. Substitution of $\gamma = \gamma_{t,0} + \eta$ yields $k+1$ inequality restrictions on η that restrict Ω .

B5. Let $VQ_t = \{(p_t, y_t, q_t)\}$, with $p_t > 0$ and $y_t = p_t' q_t$. Solving γ from (21) yields

$$\gamma_{it} = q_{it} - \lambda \alpha_i / p_{it} \quad (i=k+1, \dots, n) \text{ for some arbitrary } \lambda > 0. \quad (22)$$

The solution is feasible if $\gamma_i < q_i$ ($i=1, \dots, k$).

Ω has to be large enough to contain at least one value of η such that $\gamma_t = \gamma_{t,0} + \eta$ satisfies (22) and is feasible.

Note that this condition is similar to B4 for LES. It is weaker than B4 since the quantities q_1, \dots, q_k do not have to be rationalized.

Example 2: Quadratic Direct Utility Function (QDU)

The direct utility function is given by

$$U(q) = \gamma' q - 1/2 q' B q, \quad (23)$$

where $\gamma = (\gamma_1, \dots, \gamma_n)'$ and $B = \begin{bmatrix} \beta_{11} & \dots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \dots & \beta_{nn} \end{bmatrix}$ is positive definite.

The utility function has a satiation point at $q = B^{-1}\gamma$, with corresponding utility level $u = \frac{1}{2}\gamma'B^{-1}\gamma$. The demand functions are given by

$$q = B^{-1}\gamma - (p'B^{-1}p)^{-1}\{\gamma'B^{-1}p - y\} B^{-1}p$$

and the indirect utility function is thus given by

$$v(p, y) = 1/2 \{\gamma'B^{-1}\gamma - (p'B^{-1}p)^{-1}[\gamma'B^{-1}p - y]^2\}.$$

The indirect utility function is increasing in y as long as the satiation point is not in the budget set, i.e. as long as $y < \gamma'B^{-1}p$. Homogeneity of degree zero is satisfied automatically.

The expenditure function is given by

$$e_{B,\gamma}(p, u) = \gamma'B^{-1}p - (p'B^{-1}p)^{1/2}[\gamma'B^{-1}\gamma - 2u]^{1/2}.$$

The Hessian of the expenditure function is

$$(\partial^2 e_{B,\gamma} / \partial p \partial p')(p, u) = (p'B^{-1}p)^{1/2}[\gamma'B^{-1}\gamma - 2u]^{1/2}[(p'B^{-1}p)^{-1}(B^{-1}p)(B^{-1}p)' - B^{-1}].$$

As was to be expected, $e(p, u)$ is only defined for $u \leq 1/2 \gamma'B^{-1}\gamma$ (i.e. for u less than or equal to the satiation level) and $p \neq 0$. It is easy to show that, since B is positive definite, the matrix

$$(p'B^{-1}p)^{-1}(B^{-1}p)(B^{-1}p)' - B^{-1}$$

is negative semi-definite and of rank $n-1$. Hence, the cost function is concave for all $u \leq 1/2 \gamma'B^{-1}\gamma$.

In what follows, we assume that there is one commodity, say the n -th, for which the price is always positive: $p_n > 0$. This suggests the following choice of $V_{B,\gamma}$:

$$V_{B,\gamma} = \{(p, y); y < \gamma'B^{-1}p, p_n > 0\}.$$

Let us consider the following stochastic specification (see, e.g., Ransom, 1987b):

$$y_t = y_{t,0} + \eta_t,$$

where $y_{t,0} = (y_{t1,0}, \dots, y_{tn,0})'$ is fixed, and $\eta_t = (\eta_{t1}, \dots, \eta_{tn})'$ is a vector of random variables with $\eta_{tn} = 0$.

The elaboration of B1, B2 and B4 is as follows.

B1. $(y_{t,0} + \eta)' B^{-1} p - y > 0$ for all t , $\eta \in Q$, and $(p, y) \in V_{\min}$.

If we define V_{\min} as for LES, this condition turns into

$$\min_{0 \leq p \leq \bar{p}} \eta' B^{-1} p \geq \bar{y} - \min_t \min_{0 \leq p \leq \bar{p}} y_{t,0}' B^{-1} p, \quad (24)$$

which defines Q . It is similar to condition (19) for LES.

B2. Inversion of the demand system for given parameter values (including y) yields shadow prices and corresponding virtual income as a function of q : $p = \lambda(y - Bq)$ and $y = p'q$, where λ can be chosen arbitrarily. The solution (p, y) is a point in $V_{B, y}$ if and only if $\lambda > 0$ and $y_n - (Bq)_n > 0$. Thus, imposition of regularity in a given region Q_{\min} in q -space yields $y_n - (Bq)_n > 0$ for all $q \in Q_{\min}$. This can be achieved by restricting the values of fixed parameters only, since we have assumed that y_n is non-random. Truncation of the distribution of η is unnecessary. If, for instance, Q_{\min} is some rectangle, 2^n linear inequality restrictions on the coefficients in B and $y_{tn,0}$ result for each individual t . The conditions to be imposed in estimation are then obtained as the intersection of the inequalities for each individual.

B4. Again, let $VQ_t = \{(p_t, y_t, q_t)\}$. Solving η from the demand functions yields

$$\eta_i = \frac{p_{ti}}{p_{tn}} y_{tn,0} - y_{ti,0} + \sum_{j=1}^n \left\{ \beta_{ij} - \frac{p_{ti}}{p_{tn}} \beta_{nj} \right\} q_{tj}$$

and Q should be big enough to contain the η 's obtained in this way for all t , and for all $y_{t,0}$ and β in the admissible parameter space.

To get some more feeling for these conditions, we look at a simple numerical example for two commodities. Let $n=2$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and assume that $y_{t,0} = y_0$, fixed and independent of t .

B1. Let V_{\min} be a rectangle in normalized (p,y) -space, i.e.

$V_{\min} = \{(p,y); 0 < v_l \leq p_1/y \leq v_u \text{ and } v_l \leq p_2/y \leq v_u\}$. Since it is assumed that p_2 exceeds 0, it is convenient to work with the normalization $p_2=1$. V_{\min} can then be written as $V_{\min} = \{(p_1,y); v_l y \leq p_1 \leq v_u y \text{ and } v_u^{-1} \leq y \leq v_l^{-1}\}$

A feasible $\gamma = (\gamma_1, \gamma_2)'$ has to satisfy $\gamma_1 p_1 + \gamma_2 - y > 0$ for all $(p_1, y) \in V_{\min}$.

Thus, γ is feasible if and only if

$$\gamma_2 > v_l^{-1} - \gamma_1, \gamma_2 > v_u^{-1} - \gamma_1 v_l / v_u \text{ and } \gamma_2 > v_l^{-1} - \gamma_1 v_u / v_l.$$

Figure 8 presents the feasible area (FA1) in (γ_1, γ_2) -space. In this example (i.e., for this choice of B) the feasible area is non-empty for every $v_u > v_l > 0$.

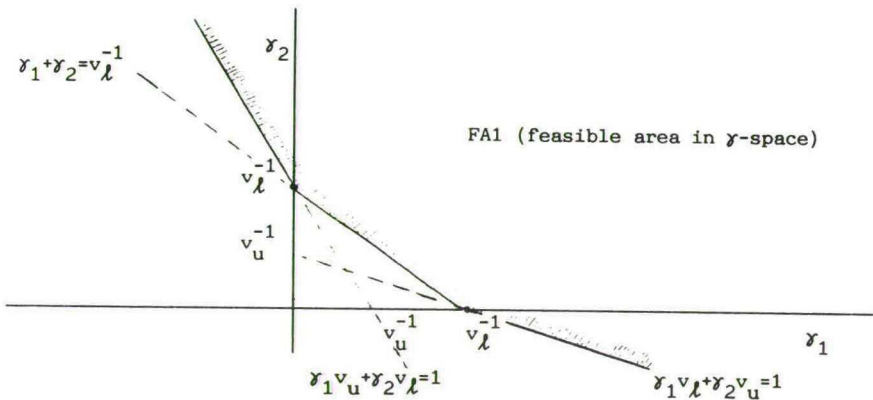


Figure 8. The feasible area in γ -space

For each feasible γ it is possible to derive the regular area in (p_1, y) -space: $V_\gamma = \{(p_1, y); \gamma_1 p_1 + \gamma_2 - y > 0\}$. The intersection of these V_γ 's is the region in (p_1, y) -space, where the indirect utility function is regular for all $\gamma \in \text{FA1}$: $V = \bigcap_{\gamma \in \text{FA1}} V_\gamma$.

In Figure 9, V and V_{\min} are sketched. Note that automatically $V \supset V_{\min}$, but the figure shows that V is larger than V_{\min} . As a matter of fact, one could have chosen V instead of V_{\min} to begin with. This would yield the same region FA1 in γ -space.

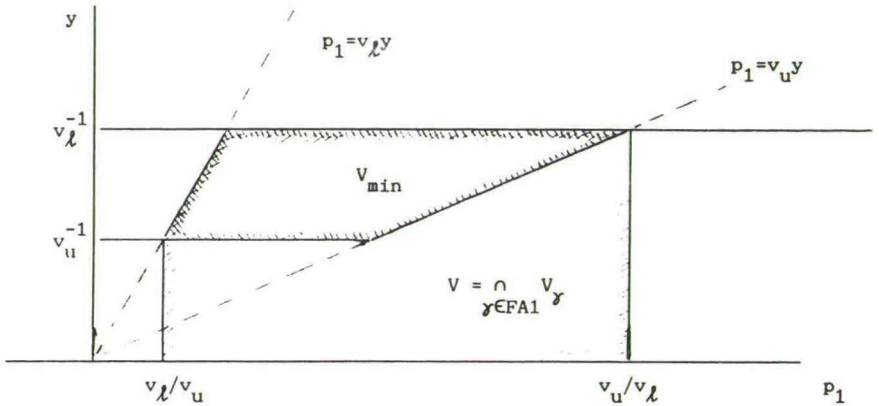


Figure 9. The minimal and the actual regular region in (p_1, y) -space

B2. For given γ , the regular region in q -space is given by

$Q_\gamma = \{(q_1, q_2) \in \mathbb{R}^2; q_2 < \gamma_2\}$. Thus, regularity on some region $Q_{\min} \subset \{(q_1, q_2) \in \mathbb{R}^2; q_2 \leq \bar{q}_2\}$ is guaranteed if γ_2 is restricted to values larger than \bar{q}_2 , and does not impose restrictions on γ_1 . The feasible area in γ -space is thus given by $\text{FA2} = \{(\gamma_1, \gamma_2); \gamma_2 > \bar{q}_2\}$.

Note that B2 cannot replace B1: The region in (p, y) -space where regularity is guaranteed for all $\gamma \in \text{FA2}$, is given by

$$V' = \cap_{\gamma \in \text{FA2}} V_\gamma = \{(p_1, y); \gamma_1 p_1 + \gamma_2 - y > 0 \text{ for all } \gamma \in \text{FA2}\} = \{(p_1, y); p_1 = 0, y \leq \bar{q}_2\}.$$

Thus, for any V_{\min} containing non-zero p_1 's, it is not possible to choose a rectangular Q_{\min} such that imposition of B2 on Q_{\min} implies that B1 holds on V_{\min} . Following the same argument, it can be shown that B1 cannot replace B2.

B4. For fixed $\gamma_2 = \gamma_{2,0}$ and given $p_1, y, (p_2=1), q_1$ and q_2 with $y = p'q$, we must find a feasible solution for γ_1 from the demand system

$$q_1 = \gamma_1 - (1 + p_1^2)^{-1} \{\gamma_1 p_1 + \gamma_2 - y\} p_1,$$

$$q_2 = \gamma_2 - (1 + p_1^2)^{-1} \{\gamma_1 p_1 + \gamma_2 - y\} p_2.$$

This is a system of two linearly dependent equations in γ_1 with a unique solution $\gamma_1 = q_1 + p_1(\gamma_2 - q_2)$. The solution is feasible if and only if

$$(p_1 + 1)\gamma_2 > v_l^{-1} - q_1 + p_1 q_2,$$

$$(p_1 + v_u/v_l)\gamma_2 > v_l^{-1} - q_1 + p_1 q_2, \text{ and}$$

$$(p_1 + v_l/v_u)\gamma_2 > v_u^{-1} - q_1 + p_1 q_2.$$

If sample prices p_1 always exceed $-v_l/v_u$, then it is possible to guarantee the existence of a feasible solution for all (q_1, q_2, y) in the sample by restricting γ_2 to be large enough.

To analyse B3 and B5, we first derive the conditional demand equations. We assume that no rationing applies to the quantity of the n -th commodity, i.e. the commodity which is treated differently from the other commodities. Solving

$$\begin{aligned} \max_{q_{II}} U_{\theta}(\bar{q}_I, q_{II}) &= (\gamma'_I, \gamma'_{II}) \begin{bmatrix} \bar{q}_I \\ q_{II} \end{bmatrix} - 1/2 (\bar{q}'_I, q'_{II}) \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{bmatrix} \begin{bmatrix} \bar{q}_I \\ q_{II} \end{bmatrix} \\ \text{s.t. } y &= p'_I \bar{q}_I + p'_{II} q_{II}, \end{aligned}$$

yields

$$q_{II} = B_{22}^{-1}(\gamma_{II} - B_{12}\bar{q}_I) - (p'_{II} B_{22}^{-1} p_{II})^{-1} [(\gamma_{II} - B_{12}\bar{q}_I)' B_{22}^{-1} p_{II} - y - p'_I \bar{q}_I] B_{22}^{-1} p_{II}, \quad (25)$$

with obvious partitioning of γ and B . The solution is feasible if (\bar{q}_I, q_{II}) is in the regular area of q -space, i.e. if $\gamma'_{II} B_{22}^{-1} p_{II} - y + \bar{q}'_I (p_I - B_{12} B_{22}^{-1} p_{II}) > 0$.

Note that (25) has the same functional form as the notional demand functions, the only difference being that y is replaced by $y - p'_I \bar{q}_{II}$, γ by $(\gamma_{II} - B_{12}\bar{q}_I)$, B by B_{22} , and p by p_{II} .

The elaboration of conditions B3 and B5 is similar to the LES case:

B3. Let $VQ_t^I = \{(p_t, y_t, q_{It})\}$.

Existence of a feasible solution for given $\gamma = (\gamma_I, \gamma_{II})$ means

$$\gamma'_{II} B_{22}^{-1} p_{II} - y_t + \bar{q}'_{It} \{p_{It} - B_{12} B_{22}^{-1} p_{II}\} > 0.$$

Substitution of $y = y_{t,0} + \eta$ yields an inequality restriction on η that restricts the set Q .

B5. Let $VQ_t = \{(p_t, y_t, q_t)\}$, with $p_{nt} > 0$ and $y_t = p_t' q_t$. Solving y from (25) yields

$$y_{II} = B_{12} q_{It} + B_{22} q_{II t} - \lambda_t p_{II t}, \quad (26)$$

where

$$\lambda_t = \{-y_{tn,0} + [B_{12} q_{It} + B_{22} q_{II t}]_n\} / p_{tn}. \quad (27)$$

The solution is feasible if and only if $\lambda_t > 0$.

Q has to be large enough to contain at least one value of η such that $y_t = y_{t,0} + \eta$ is feasible and satisfies (26), with λ given by (27).

□

Example 3: Almost Ideal Demand System (AIDS)

Let $v = (\log p_1, \dots, \log p_n)'$, $\alpha = (\alpha_1, \dots, \alpha_n)'$, $\beta = (\beta_1, \dots, \beta_n)'$, Γ a symmetric $n \times n$ -matrix with typical element γ_{ij} . The expenditure function is

$$e_{\alpha, \beta, \Gamma}(p, u) = \exp \{a(p) + u b(p)\}, \quad (28)$$

where $a(p) = \alpha_0 + \alpha'v + 1/2 v' \Gamma v$, and $b(p) = \exp(\beta'v)$.

The expenditure function is homogeneous in p if $\alpha' \iota = 1$, $\Gamma \iota = 0$ and $\beta' \iota = 0$, where ι is an n -dimensional vector with unit elements.

The uncompensated demands are given by

$$s = \alpha + \Gamma v + \beta \{\log y - a(p)\}, \quad (29)$$

where $s = (s_1, \dots, s_n)'$, s_i being the i -th budget share.

The concavity condition for the expenditure function is

$$C \text{ is negative semi-definite,} \quad (30)$$

where $C = \Gamma + \beta \beta' \{\log y - a(p)\} - \Delta + s s'$, with $\Delta = \text{diag}(s)$. Sufficient conditions for (30) are

- (a) Γ is negative semi-definite;
- (b) $\log y \leq a(p)$;
- (c) $0 \leq s_i \leq 1$ ($i=1, \dots, n$).

(31)

random preferences are introduced by defining $\alpha_t = \alpha_{t,0} + \eta_t$, with $\eta_t' t = 0$, η_t random.

B1. Suppose V_{\min} is defined by $V_{\min} = \{(p, y); 0 \leq v \leq \bar{v}, 0 \leq y \leq \bar{y}\}$.

Condition (31b) requires the following restriction on fixed parameters and on the range of the random variables:

$$\max_{0 \leq v \leq \bar{v}} (-\eta' v) \leq \alpha_0 - \log \bar{y} + \min_t \min_{0 \leq v \leq \bar{v}} (\alpha_{t,0}' v + 1/2 v' \Gamma v).$$

We also have to impose (31c). Rewrite (29) as follows:

$$s = (I - \beta v') \eta + \alpha_{t,0} + \Gamma v + \beta \{ \log y - \alpha_0 - \alpha_{t,0}' v - 1/2 v' \Gamma v \}. \quad (32)$$

From this expression it is clear that (c) imposes a number of additional linear restrictions on the range of η 's. Thus, as with LES and QDU, we find that the η 's are confined to a polyhedron, although in this case the polyhedron is more difficult to characterize.

B2. The characterization of the area in quantity space where the system is regular appears to be extremely difficult. The reason is that the regularity conditions are essentially formulated in terms of prices and income or utility level. Regularity conditions in q-space can be obtained using the inverse demand system, but in case of AIDS no explicit expression for the inverse demand system (i.e. the shadow prices) can be given. In terms of budget shares, such a characterization is substantially more straightforward. For some purposes this may suffice. Otherwise, it may be possible to rely on numerical tools.

B4. Let $VQ_t = \{(p_t, y_t, q_t)\}$ with $p_t' q_t = y_t$, $p_t > 0$ and $y_t > 0$. η can be solved from the linear system

$$s_t = [I - \beta v_t'] \eta + \alpha_{t,0} + \Gamma v_t + \beta \{ \log y_t - \alpha_0 - \alpha_{t,0}' v_t - \frac{1}{2} v_t' \Gamma v_t \}, \quad (33) \\ \eta' t = 0,$$

where $s_t = (s_{t1}, \dots, s_{tn})'$, $v_t = (v_{t1}, \dots, v_{tn})'$, $s_{ti} = p_{ti} q_{ti} / y_t$, $v_{ti} = \log p_{ti}$ ($i=1, \dots, n$). The solution must satisfy condition (30). If $0 \leq s_{ti} \leq 1$ and Γ

is negative semi-definite, then a sufficient condition for this can be derived from (29):

$$-\eta'v_t \leq \alpha_0 - \log y_t + \alpha_{t,0}'v_t + \frac{1}{2} v_t' \Gamma v_t. \quad (34)$$

Substituting the solution for η obtained from (33) in (34) yields an intricate condition on (p_t, y_t, q_t) . The fixed parameters must be chosen so that this condition is satisfied for all t .

In order to analyse the conditional demand system, note that $F_{g(\psi, \eta)}((\tilde{p}_I, p_{II}), \tilde{y}) = q$ can be rewritten as

$$\tilde{s} = [I - \beta \tilde{v}'] \eta + \alpha_{t,0} + \Gamma \tilde{v} + \beta \{ \log \tilde{y} - \alpha_0 - \alpha_{t,0}' \tilde{v} - \frac{1}{2} \tilde{v}' \Gamma \tilde{v} \}, \quad (35)$$

where $\tilde{v} = (\tilde{v}_I', v_{II}')'$, $\tilde{v}_I = (\tilde{v}_1, \dots, \tilde{v}_k)'$, $\tilde{v}_i = \log \tilde{p}_i$ ($i=1, \dots, k$), $\tilde{s} = (\tilde{s}_I', s_{II}')'$, $\tilde{s}_I = (\tilde{s}_1, \dots, \tilde{s}_k)'$, $\tilde{s}_i = \tilde{p}_i q_i / \tilde{y}$ ($i=1, \dots, k$). It is not possible to derive an analytical expression for shadow prices \tilde{p}_I from (35). Numerical methods have to be used. As a consequence, the elaboration of conditions B3 and B5 seems extremely difficult.

B3. Because of the intricate way in which \tilde{p}_I enters (35), this condition cannot be analysed analytically. In specific examples, for given values of the fixed parameters, numerical methods might prove useful, but imposition a priori seems to be impossible.

B5. Let $VQ_t = \{(p_t, y_t, q_t)\}$ as before. The condition states that (35) (with (p, y, q) replaced by (p_t, y_t, q_t)) must yield at least one feasible solution for (η, \tilde{p}_I) . Since (35) does not permit an analytic solution for \tilde{p}_I , this condition can only be checked numerically, but not imposed in any obvious way. Note again that it is weaker than B4 because the quantities \tilde{q}_I do not have to be rationalized.

□

6. Demand Systems with Endogenously Switching Regimes

In this section we consider the problem of an individual who maximizes utility subject to a set of linear inequality constraints. Common examples are the case of non-negativity constraints (see, e.g., Wales and Woodland, 1983, Lee and Pitt, 1986, Ransom, 1987a, Van Soest and Kooreman, 1987) and the kinked budget set in labour supply models (Hausman (1981, 1985), Moffitt, 1986, Blomquist, 1983, MaCurdy et al., 1988). In contrast to the discussion in the previous section, we now assume that it is not known in advance which constraints are binding and which are not. The 'regime', i.e. the way constraints are split up between those which are binding and those which are not binding, is therefore endogenous.

The utility maximization problem in its primal form can be written as

$$\max_q U_{\theta}(q) \text{ s.t. } Rq \leq r. \quad (36)$$

Here k is the number of restrictions, including the budget constraint, R is a $k \times n$ -matrix and $r \in \mathbb{R}^k$. To simplify notation we assume that R and r are fixed, but this is not essential. R and r may for instance depend on the fixed and random parameters θ . In what follows, we assume that $\{q \in \mathbb{R}^n; Rq \leq r\}$ is compact and non-empty. Specific choices of R and r yield the examples referred to above:

Example a: non-negativity constraints: $q \geq 0$,

budget constraint: $p'q \leq y$.

So $k=n+1$, $R=(p, -I)'$ and $r=(y, 0, \dots, 0)'$.

□

Example b: kinked budget constraint: $c \leq w_j h + y_j$ ($j=1, \dots, m$),

time constraints: $h \geq 0$ and $h \leq T$.

(notation as in section 2, with $q=(h, c)'$)

So $k=m+2$, $R' = \begin{bmatrix} -w_1 & \dots & -w_m & -1 & 1 \\ 1 & \dots & 1 & 0 & 0 \end{bmatrix}$, $r = (y_1, \dots, y_m, 0, T)'$.

□

If the utility function is strictly quasi-concave and continuously differentiable on the convex set $\{q \in \mathbb{R}^n; Rq \leq r\}$, then the solution of the maximization problem is unique and can be found using the following Kuhn-Tucker conditions:

q is optimal if and only if there is some $\tilde{\lambda} \in \mathbb{R}^k$ such that

$$\begin{aligned}\tilde{\lambda} &\geq 0, \\ Rq &\leq r, \\ \tilde{\lambda}'(Rq-r) &= 0, \text{ and} \\ (\partial U_{\theta}/\partial q)(q) &= R'\tilde{\lambda}.\end{aligned}\tag{37}$$

If, in addition to the conditions mentioned above, non-satiation is imposed, then (37) can be rewritten in terms of the corresponding (homogeneous of degree zero) demand system $F_{\theta}(p, y)$. This demand system has the properties

$$\begin{aligned}(\partial U_{\theta}/\partial q)(F_{\theta}(p, y)) &= \mu p \text{ for some } \mu > 0, \text{ and} \\ p'F_{\theta}(p, y) &= y.\end{aligned}$$

Making use of these properties and substituting $\lambda = \tilde{\lambda}/\mu$, (37) can be written as

$$\begin{aligned}\lambda &\geq 0, \\ Rq &\leq r, \text{ and} \\ q &= F_{\theta}(R'\lambda, r'\lambda).\end{aligned}\tag{38}$$

(Since the demand system is homogeneous of degree zero and $\lambda \neq 0$ (non-satiation), some normalisation on λ may be added). $R'\lambda$ and $r'\lambda$ can be interpreted as a vector of shadow prices and corresponding shadow income, respectively.

To illustrate the general nature of (38), we elaborate it for the two examples given above.

Example a (continued)

(38) yields $\lambda \geq 0$, $p'q \leq y$, $-q \leq 0$, and

$$q = F_{\theta}(\{\lambda_1 p - (\lambda_2, \dots, \lambda_{n+1})'\}, \lambda_1 y).$$

If the utility function is increasing in at least one of the quantities, then the budget constraint is binding, so $\lambda_1 > 0$. We can then choose the normalisation $\lambda_1 = 1$ and this yields, with $\tilde{\lambda} = (\lambda_2, \dots, \lambda_{n+1})'$:

$$\tilde{\lambda} \geq 0, p'q = y, q \geq 0, \text{ and } q = F_{\theta}(p - \tilde{\lambda}, y).$$

Thus we find the well-known result that shadow prices ($p - \tilde{\lambda}$) cannot exceed real prices (p).

□

Example b (continued)

(38) yields $\lambda \geq 0$, $c \leq w_j h + y_j$ ($j=1, \dots, m$), $-h \leq 0$, $h \leq T$, and

$$(h, c)' = F_{\theta} \left(\left(- \sum_{j=1}^m w_j \lambda_j - \lambda_{m+1} + \lambda_{m+2}, \sum_{j=1}^m \lambda_j \right)', \sum_{j=1}^m \lambda_j y_j + T \lambda_{m+2} \right)$$

Assuming that utility increases with c , $\lambda_1 + \dots + \lambda_m$ is positive, and thus λ can be normalised such that $\lambda_1 + \dots + \lambda_m = 1$. This yields

$$\lambda \geq 0, \lambda_1 + \dots + \lambda_m = 1, c \leq w_j h + y_j \quad (j=1, \dots, m), \quad 0 \leq h \leq T, \text{ and}$$

$$(h, c)' = F_{\theta} \left(\left(- \sum_{j=1}^m w_j \lambda_j - \lambda_{m+1} + \lambda_{m+2}, 1 \right)', \sum_{j=1}^m \lambda_j y_j + T \lambda_{m+2} \right).$$

If all tax-brackets consist of more than a single point, then at most two restrictions can be binding at the same time and there can only be $2m+1$ regimes: m regimes with one binding constraint and $m+1$ regimes with two binding constraints ($m-1$ kink points and two corners).

In the case of one binding constraint, say the j -th ($j \in \{1, \dots, m\}$), we have

$$(h, c)' = F_{\theta}((-w_j, 1)', y_j) \quad (\lambda_j = 1).$$

In case of a kink point, say between brackets j and $j+1$ ($j \in \{1, \dots, m-1\}$), we have

$$(h, c)' = F_{\theta}((-w_j \lambda_j - w_{j+1} [1 - \lambda_j], 1)', y_j \lambda_j + y_{j+1} [1 - \lambda_j]) = F_{\theta}((-\tilde{w}, 1)', \tilde{y}),$$

where $0 < \lambda_j < 1$. This is a familiar result: The shadow wage \tilde{w} lies between w_j and w_{j+1} and shadow income \tilde{y} satisfies $\tilde{y} + \tilde{w}h = y_j \lambda_j + w_j h = y_{j+1} + w_{j+1}h$, where h is the number of hours at the kink point. The two corners yield similar results. □

Regularity Conditions

The way in which regularity conditions for the model introduced above are formulated depends on whether we use (36), (37) or (38). If we use (36) only, then all we need is coherency of (36):

C1. For each θ , $U_{\theta}(q)$ must be defined on $\{q \in \mathbb{R}^n; Rq \leq r\}$ and (4.36) must yield a unique solution.⁵⁾

If we start from (37) and don't rely on duality results, then it is necessary to impose conditions that guarantee both coherency of (37) and

equivalence of (37) with (36) (without the latter, the micro-economic foundation of (37) would be lost). It is then sufficient to impose C2:

C2. For each θ , u_θ must be continuously differentiable and strictly quasi-concave on $\{q \in \mathbb{R}^n; Rq \leq r\}$.

According to standard Lagrange theory, C2 is sufficient for the equivalence of (36) with (37) and for coherency of (37). However, the example sketched in Figure 10 makes clear that C2 is not necessary for the latter two. In this example, u_θ is not quasi-concave on some subset of $\{q \in \mathbb{R}^n; Rq \leq r\}$, but since this subset contains no solutions of (36) nor of (37), the equivalence of the two is not affected.

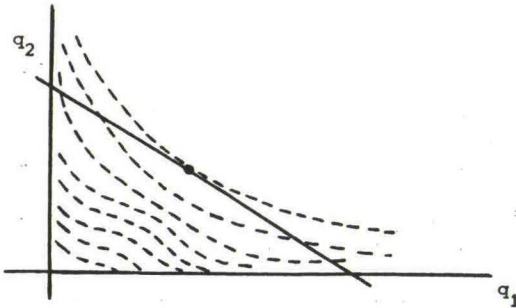


Figure 10. 'Harmless irregularity'

In order to be able to start from (38), we need an extra condition to guarantee the equivalence of (38) with (37). Substitution of $\tilde{\lambda}/\mu$ by λ is possible if non-satiation is satisfied. If F_θ is obtained from the indirect utility function or the expenditure function, an extra condition is necessary to guarantee that U_θ is defined on $\{q \in \mathbb{R}^n; Rq \leq r\}$. In terms of section 4, this means that $\{q \in \mathbb{R}^n; Rq \leq r\}$ must be contained in the regular set in q -space, $Q_{\alpha, \gamma}$. Thus it is sufficient (but, again, not necessary) to impose the condition B2, defined in section 4, with $Q_{\min} = \{q \in \mathbb{R}^n; Rq \leq r\}$:

C3. For each θ , $Q_\theta \supset \{q \in \mathbb{R}^n; Rq \leq r\}$.

Note that C3 is stronger than C2, since non-satiation on $\{q \in \mathbb{R}^n; Rq \leq r\}$ is imposed. The advantage of C3 compared to C2 is that, in principle, C3 can

also be used if no explicit specification of the direct utility function is available.

In practice, conditions C1-C3 often appear to be stronger than necessary. In Figure 10 for example, the fact that utility increases with both quantities could be used a priori to restrict the 'budget set' $\{(q_1, q_2); p_1 q_1 + p_2 q_2 \leq y, q_1 \geq 0, q_2 \geq 0\}$ to $\{(q_1, q_2); p_1 q_1 + p_2 q_2 = y, q_1 \geq 0, q_2 \geq 0\}$. This implies an extra restriction and thus a different choice of R and r . If the model is modified in this way, the example sketched in Figure 10 satisfies both C2 and C3. The restrictions implied by C1 may be relaxed in a similar way. If a certain subset of $\{q \in R^n; Rq \leq r\}$ can be excluded a priori, there is no need to require that U_θ is defined on this subset. See the examples below.

External Coherency

The condition for 'external coherency' which is the appropriate substitute for conditions B4 and B5 in the case of endogenously switching regimes, requires that, in absence of measurement or optimization errors, the error structure of the model must be rich enough to explain observed optimal behaviour. Starting from (38), the condition is given by C4:

C4. ('External Coherency') Let RQ_t be a given set of restrictions (including the budget constraint) and quantities that satisfy these restrictions. For all t , ψ and $(R, r, q) \in RQ_t$, there must be some $\eta \in Q$ such that $\lambda \in R^k$ exists with $\lambda \geq 0$ and $q = F_{g_t}(\psi, \eta)(R'\lambda, r'\lambda)$.

Similar conditions can be formulated starting from (36) or (37). Essentially, C4 is the same as B4 and B5. It implies that the support Q of the random preference terms η must be large enough. Operationalization of this condition for a given demand system may be difficult. Imposition of C4 can be avoided by explicit incorporation of measurement or optimization errors, but this is often undesirable from an economics viewpoint.

Examples

The conditions C1-C3 introduced above will be illustrated by some examples. We focus on non-negativity constraints and kinked budget sets, and use the specifications of preferences already considered in section

5. We also briefly discuss the relevance of Van Soest and Kooreman (1987) in the framework of this section.

Example 1: LES

The direct utility function corresponding to (17) can be derived by inverting (18) and substituting the resulting expression for (p, y) into (17):

$$u_g(q) = c \prod_{i=1}^n (q_i - \gamma_i)^{\alpha_i}, \text{ where } c = \prod_{i=1}^n \alpha_i^{\alpha_i}.$$

u_g is defined on $Q_{\alpha, \gamma} = \{q \in \mathbb{R}^n; q > \gamma\}$. On this region, u_g is also strictly quasi-concave. Thus, for this example, C1, C2, and C3 are equivalent and can be written as

For each γ , $\gamma < q$ for each q with $Rq \leq r$.

Bearing in mind that the condition must hold for each individual t , we have omitted the subscripts t . For the stochastic specification considered in section 5, the condition above implies, for given systematic part γ_0 of γ , truncation of the distribution of η . In case of non-negativity constraints as well as in case of a kinked budget constraint, this leads to imposition of negativity of the γ_i 's.⁶⁾

In practice however, the fact that for LES u_g is not defined in each q with $Rq \leq r$, is not considered to be a problem. The γ_i 's are interpreted as subsistence levels and $\mathbb{R}^n \setminus Q_{(\alpha, \gamma)}$ is interpreted as a region where utility is extremely low. Thus, maximization of u_g on $\{q \in \mathbb{R}^n; Rq \leq r\}$ should be interpreted as maximization of u_g on $\{q \in \mathbb{R}^n; Rq \leq r \text{ and } q > \gamma\}$. This can be incorporated from the start by including the extra restrictions $q > \gamma$ in R and r .⁷⁾ Since $\{q \in \mathbb{R}^n; Rq \leq r \text{ and } q > \gamma\} \subset Q_{\alpha, \gamma}$, the only condition which remains is

For each γ , $\{q \in \mathbb{R}^n; Rq \leq r \text{ and } q > \gamma\}$ is non-empty.

In the case of non-negativity constraints, this condition can be rewritten as

For each γ , $p' \gamma^+ < y$, where $\gamma^+ = (\gamma_1^+, \dots, \gamma_n^+)'$ with $\gamma_i^+ = \max(0, \gamma_i)$.

Note that this condition is much weaker than the condition $\gamma < 0$. In the case of kinked budget constraints, a similar result can be obtained.⁸⁾ □

Example 2: DQU

The direct quadratic utility function is defined in each point of R^n by (23). C1 holds if (36) has a unique solution. We assume that the set $\{q \in R^n; Rq=r\}$ is compact, so existence of the solution is guaranteed, and only uniqueness may be a problem. Sufficient, but not necessary, for uniqueness is strict quasi-concavity of the direct utility function, i.e. the matrix B of fixed parameters must be positive definite. For the case of non-negativity constraints, Ransom (1987a) proves coherency of (37) directly, by demonstrating that for this specification (37) can be written as a linear complementarity problem.

The fact that the choice set may contain the satiation point $B^{-1}\gamma$, in which case duality results are no longer valid, is no problem if we use (37) only. It does become a problem if we start from (38). Thus, for condition C3 to be satisfied, we need both that B is positive definite and that $B^{-1}\gamma \notin \{q \in R^n; Rq=r\}$.

Sufficient conditions for this can be derived if assumptions similar to those in section 5 are made. In particular, let us assume that in the relevant region of q-space, utility increases as a function of q_n , i.e. $\gamma_n - (Bq)_n > 0$. This leads to imposition of condition B2 for QDU with $Q_{\min} = \{q \in R^n; Rq=r\}$:

$$\gamma_n > \max_q \{(Bq)_n; Rq \leq r\}. \quad (39)$$

The maximum of the right hand side of (39) can be found by linear programming.

In the special case of non-negativity constraints, assuming that all prices are strictly positive, (39) yields

$$\gamma_n > y \max_{1 \leq j \leq n} (\beta_{nj}/p_j).$$

In case of the kinked budget constraint (39) yields

$$\gamma_n > \max_{0 \leq j \leq m} (\beta_{21}h_j + \beta_{22}c_j),$$

where (h_j, c_j) ($j=1, \dots, n$) are the corners $(h_0, c_0) = (0, y_1)$ and $(h_m, c_m) = (T, w_m T + y_m)$ and the kink points $(h_j, c_j) = (\{y_{j+1} - y_j\} / \{w_j - w_{j+1}\}, w_j h_j + y_j)$ ($j=1, \dots, m-1$).

□

Example 3: Indirect Translog

The indirect translog specification was introduced in section 2. In general, it is not possible to derive analytical expressions for shadow prices for this specification. As a consequence, an analytical expression for Q_0 , the regular region in q -space, cannot be given. We only discuss the special case of non-negativity constraints, for which there is a way to avoid these problems. In this case shadow prices corresponding to the optimal quantity vector are either real prices (if $q_i > 0$ then $\tilde{p}_i = p_i$), or can be obtained from a system of linear equations (see e.g. Lee and Pitt, 1986).

The way in which coherency can be imposed in this example was discussed in Van Soest and Kooreman (1987). By rewriting (38) as a linear complementarity problem, sufficient conditions for coherency of (38) are derived which imply restrictions on fixed parameters, without truncation of the distribution of η . These conditions are not as strong as C3, since they do not necessarily imply regularity of the demand system on $\{q \in \mathbb{R}^n; q \geq 0\}$. However, as is shown in Proposition 2 in Van Soest and Kooreman (1987), they do imply regularity at the optimum. \square

Example 4: AIDS

In the previous sections it has been shown that regularity conditions in some region Q_{\min} in q -space for this demand system are very intricate because shadow prices cannot be derived in closed form. Thus, in general, no analytical results can be derived.

For the special case of non-negativity constraints, it is possible to rewrite (38), following the procedure of Van Soest and Kooreman (1987), as a quadratic complementarity problem. However, contrary to the linear complementarity problem, we do not know of any results in the literature on the number of solutions of the quadratic complementarity problem. \square

7. Conclusions

In this paper, we have studied coherency and regularity properties of various static neoclassical models of consumer demand and labour supply. Emphasis has been placed on the relation between regularity properties of underlying preferences, in particular concavity and monotonicity, and coherency of the econometric model based on these preferences, which is to be estimated. In section 2, we have elaborated a specific example of a labour supply model with a kinked budget constraint to illustrate the

nature of the problems to be analysed. In this example, it appears that coherency is lost if and only if a specific indifference curve (for a specific value of the random parameter) is not convex at the kink point.

In section 3, we have stressed the necessity of imposing coherency conditions in practice. In an example, it is shown that even though the true data generating process is coherent, failure to impose appropriate conditions from the start may lead to inconsistent estimation of the parameters, at least if maximum likelihood is used. These estimates would then make us conclude that the model is misspecified. This also illustrates the fact that it is not possible to test whether the model is coherent or not. The requirement of coherency is after all a logical and not an empirical one.

A usual practice is to estimate the parameters by maximum likelihood without imposing coherency or regularity conditions, and then count the number of observations for which regularity is violated, given the parameter estimates. If there are observations for which regularity does not hold, this leads to the conclusion that not everyone behaves according to the neoclassical assumptions underlying the model. The example in section 3 shows that this conclusion is inappropriate if violation of regularity conditions also involves violation of coherency conditions. Note however that this argument works only in one direction. If, given the estimated parameter values, regularity conditions and thus coherency conditions are satisfied, then there is no reason to mistrust the estimates. In this case, although coherency was not a priori imposed, maximization of the (extended) likelihood over the extended parameter space yields the same results as maximization of the (actual) likelihood over the set of parameters for which the model is coherent.

If specifications for demand systems would be available which were tractable, flexible, and globally concave at the same time, then the treatment of coherency conditions would be straightforward. Given regularity of preferences, existence and uniqueness of the solution of the utility maximization problem follows from standard Lagrange theory, at least if the budget set (with all constraints taken into account) is compact and convex. In general however, tractable, flexible systems only have local concavity properties.⁹⁾ Coherency can then be guaranteed by imposing regularity conditions in some relevant region of price or quantity space.

In section 4, a general framework is sketched which shows how regularity conditions can be imposed which are relevant in case of a

standard, an inverse, or a conditional demand system. In all these cases it is a priori known which quantity restrictions are binding and which are not. We focus on systems which are specified by means of an indirect utility function or an expenditure function, since a number of flexible systems which are popular in empirical applications fall in this category. Regularity conditions generally limit the range of possible realisations of the error terms in the model (reflecting random preferences). Another condition, called 'external coherency', is introduced, which states that, on the other hand, the presence of random terms must provide so much flexibility that the data can be explained, i.e. likelihood contributions must be positive. The latter condition can be in conflict with the regularity conditions. The external coherency condition can be avoided by incorporating measurement or optimization errors, but this will often be unsatisfactory from an economics viewpoint.

The examples in section 5 show how the restrictions can be imposed in practice for a number of specifications of preferences. It appears that for LES and QDU, the restrictions can be dealt with in a reasonably tractable way. The reason for this is that explicit expressions are available for both standard demand functions and inverse demand functions, i.e., shadow prices. For a specification such as AIDS, the inverse demand equations cannot be derived analytically. As a consequence, the conditions appear to become very intricate and imposing regularity is a cumbersome affair which will only be possible with the use of numerical tools.

In section 6, we consider coherency and regularity of models with endogenously switching regimes. In this type of models, coherency is much more often a problem than in the models in section 4. Coherency conditions are derived for the linear case by Gourieroux et al. (1980), but the models based on neoclassical theory which we consider will hardly ever be linear. Instead of imposing coherency, we suggest to look at regularity conditions which are sufficient for coherency. Various examples have been given which show that resulting conditions depend on the exact formulation of the model.

Imposition of the regularity conditions considered in section 4 in the relevant area of quantity space will in general be sufficient but may be very restrictive. It may also be conflicting with the external coherency requirement. Often it is clear that less restrictive conditions will suffice, because in practice most of the budget set is irrelevant for the individual anyway. Thus it is sufficient if the utility function is quasi-concave in some part of the budget set, of which it is a priori

clear that it will contain the optimum. (we can for instance often ignore all interior points of the budget set). By making the area where regularity conditions are imposed as small as possible, maximum flexibility of the functional form is retained. At the same time this may complicate the analysis since it is often cumbersome to specify in which area regularity should be imposed.

Again, the procedure is illustrated by some examples. Among these is a brief discussion of the indirect translog with non-negativity constraints. It is explained how this fits in the framework considered in section 6. One of the flexible forms not dealt with here is the system introduced by Hausman and Ruud (1984). Regularity properties of this system and the way to impose them are discussed in Kapteyn et al. (1989). The empirical application in that paper illustrates how regularity conditions can be imposed in practice, along the lines set out in sections 4 and 6.

The approach of imposing regularity conditions a priori to guarantee coherency suggested in this paper, suffers from a number of drawbacks and complications. First of all, the conditions given are generally sufficient but only in very specific cases can it be shown that they are, in some sense, also necessary. This is important because imposition of unnecessary conditions affects the flexibility of the specification of preferences.

Another complication arises because the budget set and the parameters may differ across individuals. We have seen that certain conditions, like 'external coherency', suggest that the parameter space should not be too small, whereas other conditions suggest that it should not be too large. These conditions may easily be conflicting.

An issue that is somewhat related to the previous points, is that the stochastic specification tends to be difficult. In the examples considered the support of the random variables was usually constrained to a polyhedron. If, for instance, we would specify a normal distribution for the random preferences, this would lead to complicated truncations.

Another implication of the analysis appears to be that in most models with endogenous regimes or corner solutions, analytical expressions for the parameter restrictions implied by the regularity conditions can only be obtained if a closed form expression of the direct utility function is available. This is rather clear from the analysis in section 6, but also under exogenous rationing, conditions like B2 or B3 require knowledge of shadow prices in a rationing point. Although in principle shadow prices can be computed numerically whenever given in implicit form, it is hard to

see how conditions like B2 or B3 should be imposed when no closed form expressions for shadow prices are available. And, of course, knowing shadow prices corresponding to given quantities amounts to knowing the direct utility function. As a result, many of the popular flexible forms like AIDS or Indirect Translog cannot be used in general. In this paper we have illustrated how the various conditions can be imposed in a reasonably tractable way for LES and QDU, systems for which the direct utility functions are available. In Kapteyn et al. (1989), it is shown that a similar procedure is possible for the Hausman Ruud specification.

Altogether, the treatment of endogenous regimes or corner solutions appears to require rather intricate procedures for the imposition of regularity conditions and it severely limits the number of functional forms that can be considered. Despite these difficulties, it should be clear that without the imposition of regularity conditions one will often end up with a nonsensical model. Thus the choice appears between complexity and incoherency.

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Notes

- 1) The support of an absolute continuous probability distribution with probability density function f is defined as the closure of $\{x; f(x) > 0\}$.
- 2) Take for example $\beta=20$, $\gamma=0$, $w_1=1$, $w_2=1/2$, $y_1=1$, $y_2=11$, $T=40$, so $h_0=20$ and $c_0=21$ (the first numerical example in this section). Then (4.7') is satisfied. If, e.g., $\delta=1$, then, according to (4.5), the optimum is $h^*=h_2=21$, and $c^*=21.5$. At (h^*, c^*) , U is not quasi-concave, since (4.4) is not satisfied. In this case, the econometric model is coherent, but its micro-economic foundation is lost.
- 3) It is easily verified that the function $\text{plim}\{L(\beta_1, \beta_2, \gamma_1, \gamma_2)/T\}$ attains two local maxima on the set $\{(\beta_1, \beta_2, \gamma_1, \gamma_2); \gamma_1 \gamma_2 = 0\}$:
 $\text{plim}\{L(0.443, 0.573, 0, -0.841)/T\} = -2.26$, and $\text{plim}\{L(1, 0, -1, 1)/T\} = -2.17$.
 The global maximum is thus attained for the true parameter values $\beta_1=1$, $\beta_2=0$, $\gamma_1=-1$, $\gamma_2=0$.
- 4) In some cases, it may be useful to allow that V_{\min} depends on t . For example, it may be the case that certain values of (p, y) are only observed for individuals with specific characteristics x_t .
- 5) As in section 4.5, we assume that $\theta = g_t(\psi, \eta_t)$, where ψ is fixed and η is random with support Ω . 'For each θ ' thus should be interpreted as 'for each $\psi \in \Psi$ and for each $\eta \in \Omega$ '.
- 6) In labour supply models based on LES, leisure l is used instead of working hours h ($=T-l$), since LES is only defined for positive prices. Thus

$$u_\theta = c (l - \gamma_l)^\alpha l^{c-\gamma_c} c^\alpha.$$

The condition that 'all γ_i 's should be negative' thus should be interpreted as $\gamma_l < 0$ and $\gamma_c < 0$.
- 7) Note that in that case R and r depend on γ and thus contain a random component. Our notation does not explicitly allow for this.

- 8) With the notation of the previous footnote, the condition is:

For each $(x_L, x_C): x_L < T$ and $x_C \leq (T - \max(0, x_L))w_j + y_j$ for all $j \in \{1, \dots, m\}$.
 Given that w_j and y_j by definition imply that the budget set is non-empty, this condition is weaker than the one in the previous footnote.

- 9) For example, the generalized McFadden cost function proposed by Diewert and Wales (1987) is second order locally flexible and concave on the positive orthant of price space, but is not tractable, in the sense that it does not permit explicit expressions for Marshallian demand functions.

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